ON THE EXTENSION OF OPERATORS WITH A FINITE-DIMENSIONAL RANGE

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1. Introduction

In this note we continue the study of problems concerning extension of compact operators and the characterization of the spaces having various extension properties. In [11], [12] and [13] our main interest was in characterizing the spaces X for which certain classes of operators having X as range space can be extended. In [14] some further results concerning the *"into"* extension problem are announced. Our aim here is to complement these results by studying the corresponding *"from"* extension properties. The main result of the present note shows that if all the operators defined on X and having a 3-dimensional range can be extended in a norm-preserving manner then the same is true for every compact operator defined on X, and this is the case if and only if X^* is an L_1 space (see Theorem 1 for a precise formulation). It is shown (Theorems 2 and 3) that in the preceding results 3 cannot be replaced by 2. Indeed, all the L_1 spaces have the "from" extension property for operators with a 2-dimensional range. In Section 4 a comparison between the "into" and "from" extension properties for compact operators is given. The paper ends with two theorems concerning the lifting of operators with a 2 or 3-dimensional range. These theorems complement some results of Köthe and Grothendieck [3].

Notations. By "operator" we mean a bounded linear operator. All Banach spaces are assumed to be over the reals. Let I be a set; the Banach space of all bounded real-valued functions on I with the supremum as norm is denoted by $m(I)$. If I is finite and consists of n points we denote $m(I)$ also by l_∞^n. The conjugate of l_∞^n is denoted by l_1^n; it is the n-dimensional L_1 space. The cell $\{x; x \in X, \| x - x_0 \| \leqq r\}$ is denoted by $S_X(x_0, r)$. A Banach space X is called a $\mathcal{P}_\lambda$ space if from every $Y \supset X$ there is a projection of norm $\leqq \lambda$ onto X. For the basic facts concerning $\mathcal{P}_\lambda$ spaces we refer to the book of Day [2, pp. 94–96]. The projection constant $\mathcal{P}(X)$ of a Banach space X is defined by (cf. [4])

$$\mathcal{P}(X) = \inf \{\lambda; X \text{ is a } \mathcal{P}_\lambda \text{ space}\}.$$

For a set K in a Banach space $\mathrm{Cl}(K)$ [resp. $\mathrm{Int}(K)$] denotes the norm closure [resp. interior] of K.

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