CLOSURE PROPERTIES AND PARTIAL ENGEL CONDITIONS IN GROUPS

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1. Introduction

Let $\mathfrak{p}$ be a set of primes and σ a partial ordering of $\mathfrak{p}$ so that $p \sigma p$ is false for every prime p in $\mathfrak{p}$, and $a \sigma b$, $b \sigma c$ implies $a \sigma c$. For convenience we write $q \bar{\sigma} p$ if either $q \notin \mathfrak{p}$ or else $q \sigma p$ is false. A σ-segment is a nonempty subset $\mathfrak{s}$ of $\mathfrak{p}$ defined by the following property: if q belongs to $\mathfrak{s}$ and $p \sigma q$, then p too belongs to $\mathfrak{s}$. The group G is called $\mathfrak{a}$-closed for $\mathfrak{a}$ a set of primes, if products of $\mathfrak{a}$-elements of G are $\mathfrak{a}$-elements. A torsion group is termed $(\mathfrak{p}, \sigma)$-dispersed if it is $\mathfrak{s}$-closed for every σ-segment $\mathfrak{s}$ of $\mathfrak{p}$ (cf. [1, Definition, p. 620]). Further we say that a group G possesses a σ-minimal prime p if p belongs to $\mathfrak{p}$, if G contains elements of order p and if p is minimal among the G-relevant primes in $\mathfrak{p}$ relative to the ordering σ.

Suppose that G is a finite group and that the set $G_{\mathfrak{p}}$ of all $\mathfrak{p}$-elements of G is a $(\mathfrak{p}, \sigma)$-dispersed subgroup of G. Evidently $G_{\mathfrak{p}}$ is a characteristic subgroup of G and G possesses the following properties:

1. *If x is a p-element of G with $p \in \mathfrak{p}$ and if y is a q-element of G with $q \notin \mathfrak{p}$, then for almost every positive integer i the order of $x^{(i)} \circ y$ is divisible by primes r with $r \sigma p$ only.*
2. *Every $\mathfrak{p}$-subgroup of G is $(\mathfrak{p}, \sigma)$-dispersed.*

It is our objective to investigate whether these properties of a torsion group G are sufficient to show that $G_{\mathfrak{p}}$ is a $(\mathfrak{p}, \sigma)$-dispersed subgroup of G. In the case that G is finite we can give a positive answer. But if G is only supposed to satisfy the local double chain condition for subgroups, which may or may not be equivalent to local finiteness, we have to impose additional—possibly superfluous—conditions, because our method does not go through otherwise.

2. Results

For the terminology used in the statement of our results the reader is referred to Section 3.

THEOREM. *If σ is a partial ordering of the set $\mathfrak{p}$ of primes, and if the group G satisfies the double chain condition locally, then the following properties of G are equivalent:*

(a)
1. *$G_{\mathfrak{p}}$ is a $(\mathfrak{p}, \sigma)$-dispersed subgroup of G.*
2. *If $p \in \mathfrak{p}$, then p-factors of G are locally finite.*

Received August 30, 1963.

[1] Research supported in part by the European Research Office, U. S. Department of the Army.