INEQUALITIES FOR SUBPERMANENTS[1]

BY[2]

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I. Introduction and Statement of Results

If A is an r-square complex matrix then the permanent of A is defined by

$$\text{per}\,(A) = \sum_{\sigma \epsilon S_r} \prod_{i=1}^{r} a_{i\sigma(i)}$$

where the summation extends over the whole symmetric group S_r of degree r. This function has considerable significance in certain combinatorial problems [7, p. 24]. The problem of finding relationships between rather awkward combinatorial matrix functions such as the permanent, and the more classical algebraic invariants is one of considerable interest and importance.

In a paper in the Illinois Journal in 1957 [3] the first of the present authors obtained an upper bound for the sum of the squares of all $\binom{n}{r}^2$ r-square subdeterminants of an n-square matrix A. This work was very recently generalized and improved in an interesting paper by Ryff [6]. In the present paper we turn our attention to the substantially more difficult problem of obtaining a significant upper bound for the sum of the squares of the absolute values of all $\binom{n}{r}^2$ r-square subpermanents of an n-square complex matrix A. We then apply our main result to the case of an incidence matrix for a (v, k, λ) configuration (Theorem 3).

We shall use the following notation throughout the paper. If A has real eigenvalues, then $\lambda_1(A) \geqq \lambda_2(A) \geqq \cdots \geqq \lambda_n(A)$ will denote these. The singular values of A (defined to be the numbers $\lambda_j^{1/2}(A^*A) \geqq 0, j = 1, 2, \cdots, n$) will be designated by $\alpha_1(A) \geqq \alpha_2(A) \geqq \cdots \geqq \alpha_n(A)$. If $1 \leqq r \leqq n$, then $Q_{r,n}$ will denote the set of $N = \binom{n}{r}$ strictly increasing sequences ω, $1 \leqq \omega_1 < \omega_2 < \cdots < \omega_r \leqq n$; $G_{r,n}$ is the set of $\binom{n+r-1}{r}$ non-decreasing sequences ω, $1 \leqq \omega_1 \leqq \omega_2 \leqq \cdots \leqq \omega_r \leqq n$. If α and β are in $G_{r,n}$ then $A[\alpha \mid \beta]$ is the r-square matrix whose i, j entry is $a_{\alpha_i \beta_j}$, $i, j = 1, 2, \cdots, r$. If $a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$ is any set of n non-negative numbers then there are $\binom{n+r-1}{r}$ homogeneous products $a_\omega = \prod_{t=1}^{r} a_{\omega_t}$, $\omega \epsilon G_{r,n}$. Now, although $a_1^r \geqq a_1^{r-1}a_2$ are the two largest of these products, it is not true generally that the ordering according to magnitude and the lexicographic ordering of the a_ω, $\omega \epsilon G_{r,n}$ coincide (e.g. $a_1^{r-2}a_2^2$ is not necessarily smaller than $a_1^{r-1}a_3$). We let $L_r(a_1, a_2, \cdots, a_n)$ designate the sum of the largest $N = \binom{n}{r}$ of the $\binom{n+r-1}{r}$ homogeneous products a_ω.

Received June 26, 1963.

[1] The research of both authors was supported by the National Science Foundation.

[2] The authors would like to take this occasion to make the following correction in their paper *Generalizations of some combinatorial inequalities* by H. J. Ryser, this journal, vol. 7 (1963), pp. 582–592: On page 591, line 19, instead of "The matrix PP^T is nonnegative," read "The matrix $PP^T = H$ is nonnegative."