GENERALIZED LINEAR DIFFERENTIAL SYSTEMS AND RELATED RICCATI MATRIX INTEGRAL EQUATIONS

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1. Introduction

In a previous paper [7], the author considered a generalized differential system that was equivalent to a type of linear vector Riemann-Stieltjes integral equation, and which included as special cases the real scalar generalized second-order differential equations occurring in the works of Sz.-Nagy, ([10] and [9, pp. 247–254]), and Feller [3], and also certain systems with "interface" conditions associated with the accessory differential equations for a simple integral non-parametric variational problem. The considered systems, however, did not include such modified accessory systems for variational problems of Lagrange or Bolza type. Moreover, the treatment of [7] was limited to first order systems of order $2n$, in which the component vector admitting possible discontinuities was of dimension n. The present paper deals with a generalized differential system whose form is inclusive enough to remove these two limitations. Moreover, attention is focused on the central role played by a non-linear matrix integral equation of Volterra type, which will be referred to as a "Riccati matrix integral equation" in view of its intimate relation to the Riccati matrix differential equation. Finally, for generality, and also for application in the "control formulation" of certain variational problems, the results are presented in a form which for accessory systems under classical variational assumptions, (see, for example, [1, §§39, 81]), would be in terms of canonical variables, but which far exceeds this particular instance.

The basic relationships between the considered generalized matrix differential system and certain functionals are derived in §2, while §3 is concerned with properties of such a system and its adjoint. The fundamental connections between such systems and Riccati matrix integral equations are presented in §4. The important instance of self-adjoint systems is treated in §5, with particular attention to the interrelations between criteria of non-oscillation, the existence of solutions of associated Riccati matrix integral equations, and the positive definiteness of certain hermitian functionals. In §6 some of the results of §5 are applied to a special scalar integral equation, concerning which Cameron [2] initially raised a question of solvability that was answered by Woodward [11]. Finally, in §7 there are presented two theorems that extend earlier results of Hestenes [3], Bliss [2, §87] and the author [6] on accessory systems for variational problems of Bolza type.

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