HOMOTOPY GROUPS OF THE SPACE OF HOMEOMORPHISMS ON A 2-MANIFOLD

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1. Introduction

This is the final paper in a series of papers concerning the homotopy groups of the space of homeomorphisms on a 2-manifold. If M is a compact 2-manifold with boundary $M^{\cdot}$, and K is a closed subset, denote by $H(M, K)$ the space of homeomorphisms of M onto itself leaving K pointwise fixed and by $H_0(M, K)$ its identity component. Kneser proved [14] that the space of rigid motions on S^2 is a deformation retract of $H_0(S^2)$. Thus $\pi_n H_0(S^2) = \pi_n(P^3)$ for each n, $\pi_n H_0(S^2) = \pi_n(S^3)$ for $n > 1$, and $\pi_n H_0(S^2) = \pi_n(S^2)$ for $n > 2$. In particular $\pi_1 H_0(S^2) = Z_2$ and $\pi_n H_0(S^2) = 0$. If M is a disc with holes or a Moebius strip, $H_0(M, M^{\cdot})$ is homotopically trivial ([6], [8] and [12]). In fact Alexander's classic result [1] that the space of homeomorphisms of an n-cell onto itself leaving the boundary pointwise fixed is contractible and locally contractible is a most important tool in the study of these problems. If M is a torus, $\pi_i H_0(M) = \pi_i(M)$ for each i, and if M is a torus with the interiors of a finite number of disjoint discs removed, $H_0(M, M^{\cdot})$ is homotopically trivial [11]. For real projective space, $\pi_i H_0(P^2) = \pi_i(P^2)$ for $i > 2$, $\pi_2 H_0(P^2) = 0$, $\pi_1 H_0(P^2) = Z_2$, $\pi_1 H_0(P^2, x) = Z$, where $x \in P^2$ and $\pi_i H_0(P^2, x) = 0$ for $i > 1$ (see [12]). For the Klein bottle K, $\pi_i H_0(K) = 0$ for $i > 1$, $\pi_1 H_0(K) = Z$ and $\pi_i H_0(K, x) = 0$ for each i [12]. In this present paper, it is shown that $H_0(M)$ is homotopically trivial for all compact 2-manifolds (without boundary) of genus greater than 1, if orientable, and greater than 2, if non-orientable; and that, if M is a compact 2-manifold with non-empty boundary, $H_0(M, M^{\cdot})$ is homotopically trivial.

Further related results may be found in McCarty's paper [16], where he proves among other things, that

$$\pi_1 H_0(S^2, x) = \pi_1 H_0(S^2, x \cup y) = Z$$

and

$$\pi_i H_0(S^2, x) = \pi_i H_0(S^2, x \cup y) = 0 \qquad \text{for} \quad i > 1$$

and that if K is a finite subset of S^2 with more than two points $H_0(S^2, K)$ is homotopically trivial. Quintas proves in [18] that if M is an orientable compact manifold with two or more handles or is non-orientable with three or more cross-caps and M_k is the manifold obtained from M by deleting k points, $\pi_n H_0(M) = \pi_n H_0(M_k)$ for each n. It thus follows from McCarty's work that in this case $\pi_n H_0(M) = \pi_n H_0(M, x)$.

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