CO-EQUALIZERS AND FUNCTORS

BY

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0. Introduction

If X and Y are objects of a category $\mathbf{C}$, let $| X, Y |$ denote their associated morphism set. Similarly if S and T are functors let $| S, T |$ denote the class (not necessarily a set) of natural transformations from S to T. Unless otherwise stated all functors will be assumed to be covariant. Let $R : \mathbf{V} \to \mathbf{W}$ be a functor. Then $X \in \mathbf{V}$ is a (*left*) *R-object* if for every $Y \in \mathbf{V}$ the mapping function

$$R : | X, Y | \to | RX, RY |$$

is a bijection.[1] We shall find in various circumstances certain conditions some necessary others sufficient for X to be an R-object. It is clear that such information could be of interest, however our objective is to consider the case $\mathbf{V} = \mathbf{V}(\mathbf{C}, \mathbf{D})$ a subcategory of the functor category $(\mathbf{C}, \mathbf{D})$ and $\mathbf{W} = \mathbf{W}(\mathbf{A}, \mathbf{D})$ a subcategory of $(\mathbf{A}, \mathbf{D})$ in which $R : \mathbf{V} \to \mathbf{W}$ is induced by a functor $J : \mathbf{A} \to \mathbf{C}$. Then to say that $S \in \mathbf{V}$ is an R-object means that for every $T \in \mathbf{V}$ and every $u' \in | SJ, TJ |$ there exists a unique $u \in | S, T |$ such that $uJ = u'$. The situation described arises frequently in connection with "uniqueness theorems". Thus to cite one celebrated example, if $\mathbf{V}$ is the category of homology theories on the category $\mathbf{C}$ of triangulable pairs and pair maps and if J is the functor which injects the subcategory "generated by" a single point then Eilenberg and Steenrod proved [3] that each homology theory S is an R-object in $\mathbf{V}$.

In this paper we shall be chiefly concerned with the case $\mathbf{A} = \mathbf{X}$, the subcategory of $\mathbf{C}$ consisting of a single object X and its $\mathbf{C}$-endomorphisms, J being the injection functor and we shall describe an R-object $S \in \mathbf{V}$ as an *X-functor in* $\mathbf{V}$. It follows that the X-functors are determined (up to natural equivalence in $\mathbf{V}$) by their action on $\mathbf{X}$.

In general our basic assumption is that there exists a functor $L : \mathbf{W} \to \mathbf{V}$ and a natural transformation $\alpha : LR \to 1$. L is sometimes (but not always) a left adjoint of R and then we find:

THEOREM 0.1. *If L is a left adjoint of R then X is an R-object if and only if $\alpha X \in | LRX, X |$ is an isomorphism.*

One case in which 0.1 is involved is the following. Let $\mathbf{M} = \mathbf{M}_\Lambda$ denote the

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[1] X is an R-object if and only if (X, i_{RX}) is free over RX with respect to R in the sense of A. Frei, *Freie Objekte und multiplikative structuren*, Math. Zeitschrift, vol. 93 (1966), pp. 109–141. There is some overlap in Section 1 with Frei's results. In particular Theorem 0.1 as stated is essentially not new. (See however Remark 1.1.) Our applications are quite different.