ALGEBRAS WITH THE SPECTRAL EXPANSION PROPERTY

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Introduction

Assume that A is the algebra of all completely continuous operators on a Hilbert space. If T is a normal operator in A, then T has a spectral expansion in A in the sense that $T = \sum_k \lambda_k E_k$ where the set $\{\lambda_k\}$ is the non-zero spectrum of T and $\{E_k\}$ is a corresponding set of self-adjoint projections (of course these sets are either finite or countably infinite). This is the standard spectral theorem for normal completely continuous operators (see for example, [2, Theorem 4, p. 183, and Theorem 6, p. 186]). In this paper we consider general algebras A with involution in which a spectral theorem of this type holds for every normal element in A. The formal definitions of what this means in an arbitrary algebra are given in Definitions 3.1 and 3.2. In Theorems 3.3 and 3.5 we characterize these algebras as *-subalgebras of the completely continuous operators on a Hilbert space which are modular annihilator algebras. It is a consequence of Theorem 3.3 that every semi-simple normed modular annihilator algebra A with a proper involution has the property that every normal element in A has a spectral expansion in A.

The first version of this paper was concerned only with a proof of this result. We acknowledge a debt to the referee who strengthened the original theorem and simplified its proof. In particular the proof of Lemma 2.6 is due to the referee.

1. Preliminaries

In general we use the definitions in C. Rickart's book, [4]. We assume throughout this paper that A is a complex algebra. For M a subset of A, we denote by $R[M]$ and $L[M]$ the right and the left annihilator of M respectively (that is $R[M] = \{a \in A \mid Ma = 0\}$). When A is semi-simple, A is a modular annihilator algebra if for any maximal modular left ideal M of A, $R[M] \neq 0$; the elementary properties of modular annihilator algebras are given in [1] and [7]. A subset M of A is orthogonal if whenever $u, v \in M$, $u \neq v$, then $uv = 0$.

We shall be concerned with algebras which have an involution $*$. $*$ is a proper involution if whenever $vv^* = 0$, then $v = 0$. If A has an involution $*$ and a norm $\|\cdot\|$ such that $\| vv^* \| = \| v \|^2$ for all $v \in A$, then we say that the norm $\|\cdot\|$ has the B^*-property. $u \in A$ is self-adjoint if $u = u^*$ and normal if $uu^* = u^*u$.

Now assume that A is a semi-simple modular annihilator algebra with a

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