ON CHARACTERISTIC VECTOR FIELDS[1]

BY

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Introduction

In an attempt to formulate geometrically, and to generalize, the notion of convexity introduced by Hormander in [V, Chapt. VIII] for a partial differential operator we have found it necessary to consider the characteristic vector field of a first order partial differential equation in a more general context than usual. This paper is devoted to this formulation of characteristic vector fields.

The classical notion of a characteristic vector field can be formulated in the following way. Consider a $(p + 1)$-dimensional C^∞ real manifold M and the bundle $G_p(M)$ of p-planes over M, i.e. $G_p(M)$ consists of all (m, P) where m is any point of M and P is any p-dimensional subspace of the tangent space to M at m. One makes $G_p(M)$ into a C^∞ manifold, and a bundle over M, in a natural way. There are certain natural 1-forms on $G_p(M)$ that we call lift-forms and any two of these differ only by a factor which is a C^∞ function. If F is a real-valued C^∞ function defined on an open subset U of $G_p(M)$ and if we choose any one of these lift-forms, λ, then there is a natural way to associate with F a unique C^∞ vector field V, defined on U, which is called the characteristic vector field of F, relative to λ. If a different λ were used then a different vector field would be obtained but it would only differ by a factor which is a C^∞ function. Using V one can solve easily the non-characteristic Cauchy problem for the partial differential equation defined by F, and it is for this purpose that this V was introduced.

This is one way of phrasing the classical reduction of such a partial differential equation to an ordinary differential equation. Hormander, in [V], has also used this V for expressing a convexity condition, altho he has not described V in these terms. The aim of this paper is to express and understand V in a sufficiently general context to give a geometric interpretation to Hormander's convexity condition. We shall not discuss convexity in this paper, however.

We now explain briefly the more general context in which we shall consider this V. Consider a d-dimensional manifold M, where $d > p + 1$ and the bundle $G_p(M)$ of p-planes over M. So $G_p(M)$ now contains, in some sense, "many more" p-planes than in the previous case where $d = p + 1$. Let F be a C^∞ real-valued function defined on an open subset U of $G_p(M)$. There is no notion of characteristic vector field in this case. But if one considers a $1:1$ non-singular map ϕ of $R^{p+1} \to M$ it induces, in an obvious way (via its differential) a map ϕ' of $G_p(R^{p+1}) \to G_p(M)$. The image of $G_p(R^{p+1})$ will be a sub-

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