SOME PROPERTIES OF A PARTIAL DIFFERENTIAL OPERATOR

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1. Introduction

Suppose E is a real or complex Banach space, let M denote a closed positive cone in E (possibly a subspace of E, and possibly all of E). Let S denote an open subset of E such that $S + M \subset S$, and let X denote the Banach space of all bounded and uniformly continuous complex valued functions on S. Suppose p is a continuously Fréchet differentiable function from S into M, and that p', the Fréchet derivative of p, is a bounded function. Let $D(A)$ denote the set of all Fréchet differentiable x in X such that $x'p$ is in X, and let A denote the operator in X with domain $D(A)$ defined by $Ax = x'p$. Various properties are developed for the operator A, A^2, $A + Q$, $A^2 + Q$, and $A^2 + PA + Q$, where P and Q are bounded operators in X, and the results have applications to partial differential equations. If E is real or complex Euclidean n-space, then

$$Ax = \sum p^i D_i x,$$

where p^i denotes the i^{th} component of p, and $D_i x$ denotes the i^{th} place partial derivative of x in the ordinary sense.

Most of the results require that p be a bounded function and are obtained by giving a simple formula for a strongly continuous semi-group (group in case M is a subspace of E) of operators in X which is generated by a closed extension of A. In case E is real Euclidean n-space, the generator is the minimal closed extension of A. In case M is a subspace of E, there is a simple formula for a strongly continuous semi-group generated by a closed extension of A^2. The subspace case is of no interest if E is complex, because then $D(A)$ contains only the constant functions. If E is a real Banach space, then the results can, by [3], be extended to the operators qA, qA^2, $(qA)^2$, etc., where q is a positive function in X which is bounded away from zero.

2. An ordinary differential equation

If g is a function from $S \times [0, \infty)$ or $S \times (-\infty, \infty)$ into a vector space, then g_2 denotes the second place partial derivative of g in the ordinary sense, and g_1 denotes the first place partial derivative of g in the Fréchet sense (see [1, Chapter VIII]).

2.1. THEOREM. *If s is in S, then there is only one function f from $[0, \infty)$ into S ($(-\infty, \infty)$ into S in case M is a subspace of E) such that $f(0) = s$, and $f'(t) = p(f(t))$ for all t in $[0, \infty)$ (all t in $(-\infty, \infty)$).*

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