A CHARACTERIZATION OF NON-SEGREGATED ALGEBRAS

BY
S. S. PAGE

1. Introduction

In [8] the author and R. W. Richardson show that for an algebra, (associative or Lie), all deformations are essentially deformations of the Jacobson radical. Since the Hochschild groups play a central role in deformation theory, one would expect a somewhat similar situation to prevail in regard to extensions of an algebra by a square zero ideal. Obviously, if the ideal is square zero and the extension is finite-dimensional, then the ideal must be contained in the radical of the extension. One can then view the Hochschild extensions as enlargements of the radical. This is, in a sense, just the opposite of what happens with deformations, i.e., in deformations the radical gets smaller, if anything.

The object of this paper is to show just which parts of the radical can be enlarged. The theorem, and its corollaries, of Section 2 along with a theorem of J. P. Jans [6] give a complete description of the situation for extensions by a simple module.

In what follows, A will denote a finite-dimension algebra over a field K with subalgebra S and Jacobson radical N such that $A = S + N$, K-direct, and S is K-separable. In view of the Wedderburn-Malcev theorem this is the case for a wide variety of algebras. By an A-module we will mean a two-sided A-module satisfying

$$a(mb) = (am)b \quad \text{for all} \quad a, \quad b \in A; \quad m \in M.$$

Following Hochschild [2], $C^n(A, M)$ will denote

$$\mathrm{Hom}_k(A_1 \otimes_k A_2 \otimes_k \cdots \otimes_k A_n, M)$$

where each $A_1 = A$. δ will denote the usual Hochschild coboundary operator and $H^n(A, M)$ the corresponding Hochschild groups.

2. Characterization theorem

DEFINITION. Let $f \in C^2(A, M)$ and $\delta f = 0$. Form the K-vector space $A + M$. We define the algebra B_f to have underlying vector space $A + M$ and multiplication given by

$$(a + m_1)(b + m_2) = ab + am_2 + m_1 b + f(a, b).$$

That B_f is an associative algebra is insured by $\delta f = 0$. Notice, also, that M appears as an ideal in B_f and $M^2 = 0$; hence M is contained in the radical of B_f. It is also well known that any extension of A with square zero kernel is of the above type [2].

Received November 28, 1966.