TRIVIAL LOOPS IN HOMOTOPY 3-SPHERES[1]

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In this paper we show that every homotopy 3-sphere possesses a cell-decomposition Γ which is in some respect especially simple:

THEOREM. *If M^3 is a homotopy 3-sphere then there exists a cell-decomposition Γ of M^3 with the following properties:*

(i) *Γ consists of one vertex E^0, r open 1-cells, $E_1^1, \cdots, E_r^1$, r open 2-cells, $E_1^2, \cdots, E_r^2$, and one open 3-cell E^3.*

(ii) *There exist (nonsingular, polyhedral) disks $V_1^2, \cdots, V_r^2$ in M^3 such that[2] $\dot{} V_i^2 = \bar{E}_i^1$ for all $i = 1, \cdots, r$.*

(iii) *The disks $V_1^2, \cdots, V_r^2$ may be chosen such that the connected components of $V_i^2 \cap V_j^2 - E^0$ ($i \neq j$, between 1 and r) are normal double arcs in which V_i^2 and V_j^2 pierce each other such that the interior of each double arc lies in $^\circ V_i^2 \cap {}^\circ V_j^2$, one of its boundary points lies in E_i^1, and the other one lies in E_j^1 (see Fig. 1), and such that $V_i^2 \cap V_j^2 \cap V_k^2 = E^0$ (if i, j, k are pairwise different, between 1 and r).*

It is a known fact that every closed 3-manifold M^3 possesses a cell-decomposition Γ with property (i) (this follows easily from results in Seifert-Threlfall [4], see [2, Sec. 5]). If M^3 is a homotopy 3-sphere, i.e., simply connected, then this is equivalent to the fact that the 1-skeleton $G^1 = \bigcup_{i=1}^r \bar{E}_i^1$ of Γ bounds a "singular fan" in M^3 (see [2, Sec. 6]). Now property (ii) of Γ means that G^1 is a wedge of trivial loops in M^3, and (iii) means that G^1 bounds a singular fan $\bigcup_{i=1}^r V_i^2$ which is especially simple in the sense that its single leaves V_i^2 are nonsingular.

As Bing has shown in [1] it would be sufficient for a proof of the Poincaré conjecture if one could show that every polyhedral, simple closed curve in M^3 lies in a 3-cell in M^3, or that the 1-skeleton G^1 of some cell-decomposition Γ of M^3 lies in a 3-cell in M^3. The property (ii) of Γ means that every single closed curve $\bar{E}_i^1 \subset G^1$ lies not only in a 3-cell V_i^3 (which may be obtained as a small neighborhood of V_i^2) in M^3 but moreover is unknotted in that 3-cell V_i^3. So one may hope that the above theorem could be used as a tool for proving the Poincaré conjecture or for deriving further partial results on homotopy 3-spheres.

Proof of the theorem

1. Preliminaries. We choose the semilinear standpoint as described in [3, Sec. 3], i.e., we assume for convenience that M^3 is a piecewise rectilinear

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[2] $\dot{}X$ denotes the *boundary*, $\bar{X}$ and ^-X the *closure*, and $^\circ X$ the *interior* of X.