A VARIATION OF THE TCHEBICHEFF QUADRATURE PROBLEM

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1. Introduction

The original Tchebicheff problem of quadrature was to find a formula of the form

$$\int_{-1}^{1} f(x)\,dx = B^{(n)} \sum_{i=1}^{n} f(x_i^{(n)}) \tag{1.1}$$

with real $B^{(n)}$, and real nodes $x_i^{(n)}$ in $[-1, 1]$ such that (1.1) is valid for polynomials of degree $\leq n$. It is well known [4] that Bernstein proved that the problem has a negative solution if $n \geq 10$. On the other hand the Gauss quadrature formula

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} A_i^{(n)} f(\xi_i^{(n)}) \tag{1.2}$$

is known to be valid for polynomials of degree $\leq 2n - 1$, with $A_i^{(n)} > 0$, $-1 < \xi_i^{(n)} < 1$ for all i. In a recent paper, Erdös and Sharma [2] have shown that an "intermediate" formula of the form

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{k} A_i^{(n)} f(y_i^{(n)}) + B^{(n)} \sum_{j=1}^{n-k} f(x_j^{(n)}) \tag{1.3}$$

with fixed k, real $y_i^{(n)}$, $x_j^{(n)}$, $A_i^{(n)}$ and $B^{(n)}$ cannot be valid in general for polynomials of degree $n + k$, if n is sufficiently large. It was also shown that if the degree of exactness of a formula of the form (1.3) is N (i.e. there exists a formula of the form (1.3) valid for polynomials of degree $\leq N = N(n)$) then $N(n) \leq C_k \sqrt{n}$, where C_k is independent of n.

In this paper we consider the problem of the validity (for polynomials) of a formula of the form

$$\int_{-1}^{1} f(x)\,p(x)\,dx = \sum_{i=1}^{k} A_i^{(n)} f(y_i^{(n)}) + B^{(n)} \sum_{j=1}^{n-k} f(x_j^{(n)}) \tag{1.4}$$

where $p(x)$ is a non-negative weight function. Although we give a complete solution of the problem only when $p(x) = (1 - x^2)^\alpha$, $\alpha > -1$, some of our results hold for more general weight functions. A special case, when $k = 0$ has been treated recently by L. Gatteschi [4] who proved that there exists a constant $n_0(\alpha)$ such that if $n > n_0(\alpha)$ then for the degree of exactness N of the formula

$$\int_{-1}^{1} f(x)(1 - x^2)^\alpha\,dx = B^{(n)} \sum_{j=1}^{n} f(x_j^{(n)}) \tag{1.5}$$

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