DEFECT GROUPS IN THE THEORY OF REPRESENTATIONS OF FINITE GROUPS

BY

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Dedicated to Oscar Zariski

1. Introduction[1]

Let G be a finite group. Let Ξ be an algebraically closed field. As is well known, the study of the characters of G is closely related to that of the *group algebra* $\Xi[G]$ and of its center $Z = Z(\Xi[G])$. We call Z the *class algebra* of G. We are concerned here with a further investigation of Z continuing the work in [1].

The dimension of Z as a Ξ-space is the class number $k(G)$ of G. Since we are interested in characters and related functions, we also consider the dual space $\hat{Z}$ consisting of all linear functions defined on Z with values in Ξ.

Write Z as a direct sum

$$Z = \oplus \sum B \tag{1.1}$$

of *block ideals* of Z, i.e. of indecomposable ideals of Z. This decomposition (1.1) corresponds to the decomposition

$$\hat{Z} = \oplus \sum F_B \tag{1.2}$$

where F_B is the subspace of $\hat{Z}$ consisting of those $f \in \hat{Z}$ which vanish on all block ideals $B_1 \neq B$ in (1.1). Then B and F_B are themselves dual vector spaces and they have the same dimension k_B.

Each B is a commutative ring with a unit element η_B. If 1 is the unit element of Z, we have

$$1 = \sum_B \eta_B \tag{1.3}$$

and (1.3) is the decomposition of 1 into primitive orthogonal idempotents. It follows that

$$\Xi[G] = \oplus \sum_B \eta_B \, \Xi[G]$$

is the decomposition of the group algebra into (two-sided) block ideals.

Since B is indecomposable, the residue class ring $\bar{B}$ of B modulo its radical is simple and hence an extension field of finite degree of Ξ. Since Ξ was algebraically closed, $\bar{B}$ is isomorphic to Ξ. We then have an algebra homomorphism ω of B onto Ξ. Clearly, ω can be extended to an algebra homomorphism ω_B of Z onto Ξ such that ω_B vanishes for all block ideals $B_1 \neq B$ in (1.1). Thus $\omega_B \in F_B$. Conversely, it is seen at once that each non-zero algebra homomorphism of Z into Ξ coincides with ω_B for some B.

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