COMPACT AND WEAKLY COMPACT OPERATORS ON $C(S)_\beta$

BY

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In 1958, R. C. Buck [1] introduced the β or strict topology on the linear space $C(S)$ of bounded continuous functions on a locally compact Hausdorff space S. This topology is defined by the seminorms

$$P_\phi(f) = \sup\{|f(x)\phi(x)| : x \in S\} = \|\phi f\|$$

where $\phi \in C_0(S)$, the subspace of functions in $C(S)$ which vanish at infinity. Since this time several authors have studied and made use of the strict topology in various settings. One may consult [2] for more specific references. In this paper a study will be made of the compact and weakly compact linear operators on this space.

The strict topology is a complete locally convex topology which is neither barrelled, bornological nor metrizable. In fact, any of these is equivalent to the compactness of S. On the other hand the strong dual of $C(S)_\beta$ is the space $M(S)$ of bounded regular Borel measures on S as was shown in [1], and furthermore, the β and supremum norm bounded sets in $C(S)$ coincide. These two facts along with the integral representation of the continuous operators on $C(S)_\beta$ into a space $C(T)_\beta$ obtained in [8] allows us to obtain the following principal result.

Let us call an operator A on $C(S)$ into a topological vector space X compact (weakly compact) if A maps β-bounded subsets of $C(S)$ into relatively compact (weakly relatively compact) subsets of X, and call A β-compact (β-weakly compact) if A maps a β neighborhood of 0 into a relatively compact (weakly relatively compact) subset of X. It will be shown that when $X = C(T)_\beta$, then A is β-compact (β-weakly compact) if and only if A is continuous with the norm topology on $C(T)$ and compact (weakly compact). As a consequence it will be shown that these two properties coincide when X is a Banach space.

In closing this introduction the author wishes to acknowledge the aid of the referee in improving the paper, particularly with regard to the considerably shortened proofs of Corollaries 2 and 4.

Our notation will be taken from [8] and [9] and we rely on [8] for the following result.

If A is a continuous linear operator from $C(S)_\beta$ into $C(T)_\beta$ then there is a unique mapping $\lambda : T \to M(S)$, henceforth called the kernel of A, such that

$$[Af](x) = \int_S f(y)\lambda(x)(dy) = \int_S f(y)\lambda(x, dy)$$

Received December 15, 1967.