THE FIRST OBSTRUCTION TO EMBEDDING A 1-COMPLEX IN A 2-MANIFOLD

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1. Introduction

Shapiro [4] defines the first obstruction $m^n(X)$ to embedding a simplicial complex X into Euclidean n-space, taking values in $H^n(R^*X; G)$, where R^*X is the reduced deleted product of X and where $G = Z$ if n is even, and where G is the locally trivial integer sheaf, twisted by the fundamental class of the covering $RX \to R^*X$, where RX is the deleted product of X, if n is odd.This obstruction is called $\varphi^m(X)$ by Wu [6], [7], [8], and was first known to Van Kampen [5]. The vanishing of this obstruction is a necessary condition for the existence of a piecewise linear embedding of X into R^m [7]. Wu has shown that this condition is also sufficient if $m \geq 2 \dim X$ [8], while Shapiro has proved an even stronger result [4].

In the present paper we define the obstruction to embedding a finite one-dimensional complex into a two-dimensional connected oriented manifold without boundary. (The case of a non-oriented manifold is only slightly more complicated actually; the sheaf $G^*[K, f]$ has an additional twisting on it, and everything else goes through.) If $f : X \to M$ is any map, where (X, A) is a 1-dimensional simplicial pair, and if $f \mid A$ is a differentiable embedding, we define an obstruction to homotoping f, rel A, to an embedding; $\gamma(f) \in I^2(X, A; f)$. The graded functor I^* is defined in (3.5). Since $\gamma(f)$ depends only on the homotopy class, rel A, of f (cf. 3.5.1), and $\gamma(f) = 0$ if f is already an embedding (cf. 3.1.1), we have immediately that $\gamma(f) = 0$ is a necessary condition for f to be homotoped, rel A, to an embedding. This is unfortunately not sufficient, however, as we see in example (6.3).

In Section 5, we show how to compute cohomology with coefficients in sheaves over simplicial complexes by taking cochains and coboundaries; this technique is used when defining the obstruction cocycle (cf. 3.1.1).

In a later paper, we shall define the first obstruction $\gamma(f) \in I^n(X, A; f)$ to finding an embedding g, homotopic rel A to f, if f is a map from a simplicial complex X to an n-dimensional manifold M which is already an embedding on a subcomplex A. The vanishing of this obstruction will always be necessary for the existence of g, and will also be sufficient if $\dim X \leq n - 3$ and $\dim X + \dim (X - A) \leq n$. If $\dim X + \dim (X - A) > n$, there will presumably be higher obstructions; but these will certainly not help solve the special case of a 1-complex in a 2-manifold.

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