APPLICATIONS OF A COMPARISON THEOREM FOR ELLIPTIC EQUATIONS

BY

KURT KREITH

In [1] the author applied a comparison theorem of Swanson [2] to derive criteria for the positivity of the Green's function associated with second order elliptic operators. For the special case of self-adjoint operators, similar criteria were established for the positivity of the Robin's functions associated with mixed boundary conditions. The latter results were based on a comparison theorem of the author [3].

The purpose of this paper is to extend the comparison theorem of [3] to cover a class of non self-adjoint equations and to use this comparison theorem to improve substantially on Theorem 2 of [1]. It will also be shown that a variational principle and the strong maximum principle for elliptic equations can be derived from this comparison theorem.

Let L be an elliptic operator with real coefficients defined by

$$Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu \tag{1}$$

in a bounded domain $D \subset R^n$. Our comparison theorem will deal with functions $u(x)$ and $v(x)$ which are, respectively, solutions of the boundary value problems

$$Lu = 0 \text{ in } D, \qquad \partial u/\partial\nu + \sigma u = 0 \text{ on } \partial D, \tag{2}$$

and

$$Lv + pv = 0 \text{ in } D, \qquad \partial v/\partial\nu + \tau v = 0 \text{ on } \partial D. \tag{3}$$

In (2) we allow $-\infty < \sigma(x) \leq +\infty$, where $\sigma(x_0) = +\infty$ is used to denote the boundary condition $u(x_0) = 0$. Similar notation will be adhered to for (3). It is assumed that the boundary problems (2) and (3) are sufficiently regular so that certain resolvents for L and $L + pI$ can be represented as integral operators. Specifically, in the case of (2), we assume the existence of a constant K such that for $\gamma \geq K$ the boundary value problem

$$Lu + \gamma u = f \text{ in } D, \qquad \partial u/\partial\nu + \sigma u = 0 \text{ on } \partial D, \tag{4}$$

can be solved in the form

$$u(x) = \int_D G_\sigma(x, \xi; \gamma) f(\xi)\, d\xi.$$

Here $G_\sigma(x, \xi; \gamma)$ is the Robin's function for (4), having the following charac-

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