EXTENSIONS OF GROUP THEORETICAL PROPERTIES

BY
Bernhard Amberg

If N is a normal subgroup of the group G, then G is called an extension of N by G/N. If $\mathfrak{f}$ and $\mathfrak{e}$ are any group theoretical properties, the question arises in which cases every extension of an $\mathfrak{f}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group. In Section 1 we show that for "many" factor inherited properties of finite groups this requirement upon $\mathfrak{f}$ and $\mathfrak{e}$ is equivalent to apparently weaker requirements. We quote the following typical

Theorem. *If $\mathfrak{f}$ and $\mathfrak{e}$ are factor inherited properties of finite groups such that every finite elementary Abelian p-group is an $\mathfrak{f}$-group whenever the cyclic group of order p is an $\mathfrak{f}$-group then every extension of a ι $\mathfrak{f}$-group by an $\mathfrak{e}$-group is an $\mathfrak{e}$-group if and only if every product of a normal $\mathfrak{f}$-subgroup and an $\mathfrak{e}$-subgroup is an $\mathfrak{e}$-group* (*see Theorem* 1.3).

If $\mathfrak{e}$ is a saturated property (in the sense of W. Gaschütz) or if $\mathfrak{f}$ is a property of finite soluble groups, then this theorem may be improved; see Theorems 1.5 and 1.8.

In Section 2 we consider group theoretical properties $\mathfrak{f}$ and $\mathfrak{e}$ of Artinian groups such that $\mathfrak{e}$ has the following particular form. Let θ be a class of ordered pairs $(\mathfrak{x}, \mathfrak{y})$ of group theoretical properties $\mathfrak{x}$ and $\mathfrak{y}$. Then a group G is called a hyper-θ-group, if every non-trivial epimorphic image H of G possesses a non-trivial normal subgroup N such that for some pair $(\mathfrak{x}, \mathfrak{y})$ in θ, N is an $\mathfrak{x}$-group and $H/c_H N$ is a $\mathfrak{y}$-group. Let especially the class θ consist of one element only, and let $\mathfrak{u}$ denote the universal property of being a group. If $\mathfrak{a}$ denotes the class of Abelian groups, then hyper-$(\mathfrak{a}, \mathfrak{u})$ is the class of hyperabelian groups; if $\mathfrak{t}$ is the trivial class consisting of 1 only, then hyper-$(\mathfrak{u}, \mathfrak{t})$ is the class of hypercentral groups; if $\mathfrak{z}$ denotes the property of being a cyclic group, then hyper-$(\mathfrak{z}, \mathfrak{u})$ is the class of hypercyclic groups.

If the group G possesses a normal subgroup N and a subgroup U such that $G = NU$ and $N \cap U = 1$, then G is called a splitting extension of N by U. For Artinian and soluble hyper-θ-groups, the above mentioned theorem may be improved as follows.

Theorem. *If $\mathfrak{f}$ and hyper-θ are factor inherited properties of Artinian and soluble groups, then every extension of an $\mathfrak{f}$-group by a hyper-θ-group is a hyper-θ-group if and only if every splitting extension of an $\mathfrak{f}$-group by a hyper-θ-group is a hyper-θ-group* (*see Theorem* 2.3).

Also the requirements that hyper-θ is a saturated property of Artinian and

Received May 3, 1969.