ON THE ROGERS-RAMANUJAN IDENTITIES AND PARTIAL q-DIFFERENCE EQUATIONS

BY

George E. Andrews[1]

1. Introduction

Perhaps the easiest proof of the Rogers-Ramanujan identities is the one expounded (in two different forms) by Rogers and Ramanujan [4]. The main idea is to show that two apparently different q-series both satisfy the q-difference equation

$$f(z) - f(zq) - zqf(zq^2) = 0. \tag{1.1}$$

It is an easy matter to show that if $f(z)$ is analytic at $z = 0$ and $f(0) = 1$, then $f(z)$ is uniquely determined by (1.1). This implies that the two q-series in question are actually identical, and the Rogers-Ramanujan identities follow by specializing z.

The object of this paper is to give a proof of the Rogers-Ramanujan identities which hinges almost entirely on showing that two systems of partial q-difference equations are compatible (i.e. any set of solutions for one system is a set of solutions for the other). In the final section of the paper, we discuss the extension of this technique to other problems in the theory of partitions and q-series identities.

2. Compatible q-difference equations

Definition. Consider the systems of r equations

$$F_i(f_1(x, y), \cdots, f_n(x, y), f_1(xq, y), \cdots, f_n(xq, y), f_1(x, yq), \cdots,$$
$$f_n(x, yq), f_1(xq, yq), \cdots, f_n(xq, yq)) = 0,$$

and

$$G_j(f_1(x, y), \cdots, f_n(x, y), f_1(xq, y), \cdots, f_n(xq, y), f_1(x, yq), \cdots,$$
$$f_n(x, yq), f_1(xq, yq), \cdots, f_n(xq, yq)) = 0,$$

where $1 \leq i \leq s$, $1 \leq j \leq t$. These two systems are said to be *compatible* in case every solution set $\{f_1(x, y), \cdots, f_n(x, y)\}$ of analytic functions in x and y for one system is a solution set for the other system.

Lemma 1. *Consider the partial q-difference equation*

$$\sum_{j=0}^{r} \sum_{k=0}^{s} a_{j,k}(x, y) f(xq^j, yq^k) = b(x, y), \tag{2.1}$$

Received November 20, 1969.

[1] Partially supported by a National Science Foundation grant.