A NEW RELATION ON THE STIEFEL-WHITNEY CLASSES OF SPIN MANIFOLDS

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1. Introduction

Let all manifolds considered be n-dimensional, closed, compact, connected C^{∞} manifolds. Let $\tau_M : M \to BO$ be the classifying map for the stable tangent bundle of M. Recall that $H^*(BO)$ is a polynomial algebra on the universal Stiefel-Whitney classes, $w_1, w_2, \cdots$, where all coefficients are Z_2. Let $S \subset H^*(BO)$, then define $I_n(S, \text{geom}) \subset H^*(BO)$ to be the ideal $\bigcap_M \ker \tau_M^*$ where the intersection is taken over all n-dimensional manifolds with $S \subset \ker \tau_M^*$. Let H be an n-dimensional Poincaré algebra. There is a unique right-left A-homomorphism $\tau_H : H^*(BO) \to H$, A the Steenrod algebra (see Brown-Peterson [5, Lemma 5.1, p. 44]). Define $I_n(S, \text{alg}) \subset H^*(BO)$ to be the ideal given by $\bigcap_H \ker \tau_H$ where H runs over all n-dimensional Poincaré algebras such that $\tau_H(S) = 0$.

For the cases $S = \emptyset, \{w_1\}$, $I_n(S, \text{geom})$ corresponds to the intersection of $\ker \tau_M^*$ taken over all manifolds, and respectively, all oriented manifolds. For these two cases, Brown and Peterson show that $I_n(S, \text{geom}) = I_n(S, \text{alg})$ [5, Theorems 5.2 and 5.4, p. 45]. Clearly, one has $I_n(S, \text{alg}) \subset I_n(S, \text{geom})$ for all S. [5, p. 45] gives an example to show that equality does not always hold.

In this paper, the case where $S = \{w_1, w_2\}$ will be considered. $I_n(\{w_1, w_2\}, \text{geom})$ corresponds to the intersection of $\ker \tau_M^*$ where M runs over all n-dimensional Spin manifolds.

Let $BO\langle k\rangle$ be the $k - 1$ connective covering over BO and $\mathbf{BO}\langle k\rangle$ the connected Ω-spectrum with 0^{th} term $BO\langle k\rangle$. For $k = 0$ and 2, the bottom cohomology classes of $\mathbf{BO}\langle 0\rangle$ and $\mathbf{BO}\langle 2\rangle$ induce maps

$$\eta : H_*(X, \mathbf{BO}\langle 0\rangle) \to H_*(X) \quad \text{and} \quad \gamma : H_*(X, \mathbf{BO}\langle 2\rangle) \to H_{*-2}(X)$$

on the generalized homology [12]. In Section 2, the computation of $I_n(\{w_1, w_2\}, \text{geom})^q$ is reduced to a problem about the image of the maps η and γ for the space $X = K(Z_2, n - q)$. This reduction is a generalization to Spin of Brown-Peterson results for SO [5]. The major part of this paper is devoted to obtaining certain information about the image of η for $X = K(Z_2, 2)$. These results are stated in Section 2 and used there to prove that $I_n(\{w_1, w_2\}, \text{geom})$ is not equal to $I_n(\{w_1, w_2\}, \text{alg})$ in general. In particular, it will be shown that

$$w_7 \in I_9(\{w_1, w_2\}, \text{geom})^7 \quad \text{but} \quad w_7 \notin I_9(\{w_1, w_2\}, \text{alg})^7.$$

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