# ON THE BEHAVIOUR OF LINEAR MAPPINGS ON ABSOLUTELY CONVEX SETS AND A. GROTHENDIECK'S COMPLETION OF LOCALLY CONVEX SPACES

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The first point of this note is a simple proof of Theorem 2 which states a continuity property of linear maps with respect to absolutely convex sets and is essentially due to D. J. H. Garling [4]. In Theorem 4 the approximation theorem 16.8 of Kelley-Namioka's book [7] is sharpened by dropping the closedness assumption on the absolutely convex set on which the linear forms are to be approximated. This result is used in Theorem 5 to extend Grothendieck's well-known discussion of the completion and the completeness of the dual $X'$ of a locally convex space $X$ by admitting on $X'$ topologies $\mathfrak{T}_{\mathfrak{N}}$ of uniform convergence on classes $\mathfrak{N}$ of absolutely convex subsets of $X$ whose members need not be bounded nor closed. Finally, V. Ptak's and H. S. Collin's characterization of the completeness of $(X', \mathfrak{T}_{\mathfrak{N}})$ is carried over to these (generally not linear) topologies $\mathfrak{T}_{\mathfrak{N}}$.

The following theorem can be easily deduced from Theorem 1 of Garling [4].

THEOREM 1. *Let $X$ and $Y$ be locally convex spaces, $f$ a linear mapping from $X$ into $Y$, $\mathfrak{N}$ a collection of absolutely convex subsets of $X$, directed upwards by inclusion, whose union is absorbent. Let $f$ be continuous on each $A \in \mathfrak{N}$. Then $f$ is also continuous on the closure $\bar{A}$ of each $A \in \mathfrak{N}$.*

We wish to give a direct proof for the following special case of Theorem 1 to be applied later.

THEOREM 2. *Let $X$ and $Y$ be locally convex spaces, $f$ a linear map from $X$ into $Y$, and $A$ an absolutely convex and absorbent subset of $X$ such that $f \mid A$ is continuous. Then $f \mid \bar{A}$ is also continuous.*

*Proof.* According to a lemma of A. Grothendieck [6, p. 98] it suffices to show that $f \mid \bar{A}$ is continuous at 0. Let $V$ be a neighbourhood of 0 in $Y$. Then there is an open neighbourhood $U$ of 0 in $X$ such that

$$f(U \cap A) \subset V. \tag{1}$$

The theorem will be proved if we show

$$f(U \cap \bar{A}) \subset V + V. \tag{2}$$

If $x \in U \cap \bar{A}$ there exists a real number $\rho$, $0 < \rho < 1$, such that $\rho x \in A$. Because of the continuity of $f \mid A$ at $\rho x$ there exists $y \in U \cap A$ so close to $x$ that

$$f(\rho x) - f(\rho y) \in \rho V. \tag{3}$$

Received May 3, 1971.