ON REGULAR FUNCTIONS ON RIEMANN SURFACES[1]

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1. Introduction

If an annulus $a < |z| < b$ is mapped analytically into another annulus $a' < |z| < b'$ in such a way that the index of the images of concentric circles is equal to n $(n \neq 0)$, then its module $(2\pi)^{-1} \log b/a$ is dominated by $1/|n|$-th of the module $(2\pi)^{-1} \log b'/a'$ of the image annulus. This was proved by Schiffer [8]. A number of proofs and extensions of this result were obtained by many authors [3], [4], [6]. In particular the first author of the present paper weakened the assumption about the image domain and gave an extension by the method of the extremal metric [4]. Another interesting method of proof was given by Landau and Osserman, comparing the fluxes of harmonic functions [6].

The purpose of the present paper is to extend the above result for regular functions on arbitrary Riemann surfaces which have more than one boundary component. Let us mention the result for functions $w = f(z)$ regular on the closed annulus $a \leqq |z| \leqq b$ and such that $f'(z) \neq 0$ on the boundary. The image curves $f(ae^{2\pi it})$ and $f(be^{2\pi it})$ $(0 \leqq t \leqq 1)$ divide the w-plane into several open sets on which the indices of the image curves are constant.

Let P_m and Q_n be the respective open sets on which the index of $f(ae^{2\pi it})$ is not less than m and that of $f(be^{2\pi it})$ is not greater than n. If $P_m \neq \emptyset$ and $Q_n \neq \emptyset$ for $m > n$, then the complement of the union of the closure of P_m and Q_n consists of a finite number of domains and the module of the family of curves separating $\bar{P}_m$ and $\bar{Q}_n$ dominates $(m - n)(2\pi)^{-1} \log b/a$.

Our result will be stated for regular functions on an open Riemann surface and a regular partition of its boundary. The sets corresponding to $\bar{P}_m$ and $\bar{Q}_n$ are defined in terms of exhaustions. We will give two proofs one of which is based on the method of the extremal metric and the other is based on the comparison of the fluxes of harmonic functions. It is interesting that two different methods produce the same result.

The result has many applications. For example Hayman-Kubo's estimation of the capacity of the set of omitted values [2, 5] is generalized to Riemann surfaces. As to other applications the readers are referred to the first author's paper [4].

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