THE LEVI PROBLEM IN CERTAIN INFINITE DIMENSIONAL VECTOR SPACES

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1. Introduction

Let E be a complex vector space with a locally connected Hausdorff topology T which is at least as coarse as the finite topology T_0 on E (composed of those sets whose intersection with every finite dimensional subspace is an open set in the Euclidean topology). A complex valued function f defined on an open subset D of (E, T) is *Gâteaux differentiable* if for all $a, b \in E, f(a + ub)$ is holomorphic as a function of u in D. If in addition, f is continuous for the topology T, then f is said to be *holomorphic* in D. A *domain of holomorphy* is an open set D of (E, T) such that for every boundary point $b \in \partial D$, there exists a function f_b holomorphic on D which cannot be continued (locally) as a holomorphic function to any open neighborhood of b. If f_b can be chosen the same for every $b \in \partial D$, then D is said to be a *domain of existence.*

A function g which takes on real values in the range $[-\infty, +\infty)$ is said to be *plurisubharmonic* in an open set D of (E, T) if g is upper semi-continuous and if for all $a, b \in E$, $g(a + ub)$ is either identically $-\infty$ or a subharmonic function of u in D. We say that an open set D is *pseudoconvex* if for all $a \in E$, $-\log\, d_a(z)$ is plurisubharmonic, where

$$d_a(z) = \sup\, \{\tau : z + \lambda a \in D \text{ for all } \lambda, |\lambda| \leq \tau\}.$$

There have been attempts to characterize domains of holomorphy in infinite dimensions in terms of the characterizations given in finite dimensions. A characterization of the type Cartan-Thullen can be found for certain infinite dimensional space in [6], [7], [8]. We shall give a characterization here in terms of pseudoconvexity.

For finite dimensional spaces, the following properties are equivalent: (cf. [1], [4], [5])

(i) D is a domain of existence.
(ii) D is a domain of holomorphy.
(iii) D is pseudoconvex.

(It was Levi who first conjectured this result, which was proved by Oka.) We investigate two infinite dimensional cases for which this is still valid:

(1) for any complex vector space equipped with the finite topology T_0 ;
(2) for any separable Hilbert space

It was shown in [2] (for case (1)) and [3] (for case (2)) that every domain

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