A CHARACTERIZATION OF CERTAIN FROBENIUS GROUPS

BY

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1. Introduction

Let $\mathfrak{F}$ be a collection of groups and G a finite group. Following B. Fischer, an *$\mathfrak{F}$-set* of G is a collection D of subgroups normalized by G and generating G, such that the subgroup generated by any pair of distinct members of D is isomorphic to a member of $\mathfrak{F}$.

Let p be a fixed odd prime and D an $\mathfrak{F}$-set of the nonabelian group G, such that each member of D has order p. Fischer has shown that if $\mathfrak{F} = \{G\}$, and G is solvable, then $G/Z(G)$ is a Frobenius group [4]. He has further shown that if $\mathfrak{F}$ is the collection of Frobenius groups with cyclic kernals, then G is a Frobenius group [5].

In this paper it is shown that:

THEOREM 1. *Let $\mathfrak{F}$ be the collection of groups F with $F/Z(F)$ Frobenius of odd order. Then $G \in \mathfrak{F}$, and $Z(G)$ is generated by the centers of 2-generator D-subgroups.*

As a corollary it follows that:

THEOREM 2. *Let $\mathfrak{F} = \{F\}$ with F of odd order. Then $G/Z(G)$ is a Frobenius group of odd order.*

The restriction in Theorems 1 and 2 that F have odd order is necessary. For example if $\mathfrak{F} = \{SL_2(3)\}$ then $U_3(3)$ possesses an $\mathfrak{F}$-set. The following theorem is however true:

THEOREM 3. *Let $\mathfrak{F}$ be the collection of Frobenius groups whose kernel is an elementary 2-group. Then $G \in \mathfrak{F}$.*

The analogous theorem for $\mathfrak{F}$ the collection of groups F of order pm with $(m, 2p) = 1$, probably holds. Some progress is made in this paper toward such a result.

The proof of Theorem 3 is combinatorial. The proof of Theorem 1 is more complicated, and uses signalizer arguments. A contradiction is arrived at by showing a minimal counterexample has 2-rank at most 2, or possesses a proper 2-generated core.

Certain specialized notation and terminology is used. A *D-subgroup* of G is a subgroup H with $\langle H \cap D\rangle = H$. Given $X \leq G$, $\theta(X) = \langle X \cap D\rangle$. $\mathcal{N}(X)$ is the set of proper D-subgroups of G normalized by X, and $\mathcal{N}^*(X)$ the set of maximal elements of $\mathcal{N}(X)$. $\mathcal{N} = \mathcal{N}(1)$ and $\mathcal{N}^* = \mathcal{N}^*(1)$. $m(G)$ is the 2-rank of G. $O_\infty(G)$ is the largest normal solvable subgroup of G. $F(X)$ is the set of fixed points of X under its action by conjugation on D.

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