ON THE DIFFERENTIABILITY OF FUNCTIONS OF SEVERAL REAL VARIABLES

BY

C. P. CALDERÓN AND J. E. LEWIS

Introduction

This paper is concerned with two problems. In the first place, we show that if a function f belongs to $L_n^1(R^n)$ (see definition below), then $f(x)$ possesses total differential of order n at almost all the points of R^n (see definition below).

In the second place, if we restrict our attention to functions arising from Bessel Potentials of order n, that is

$$f(x) = \int_{R^n} G_n(x - y)g(y)\, dy$$

where $\hat{G}_n = (1 + |x|^2)^{-n/2}$, $\hat{G}_n$ is the Fourier Transform of G_n and $g \in L^1(R^n)$, our result in this case is that if $g \in L^1(R^n) \cap L^1 \log^+ L^1$ then, f possesses total differential of order n at almost all the points of R^n. This result is the best possible in the sense that given an Orlicz Class $L_\phi(R^n)$ that contains a function g for which

$$\int_{R^n} |g| \log^+ |g|\, dx = \infty,$$

then there exists a function $g_0 \in L^1(R^n) \cap L_\phi(R^n)$ such that $G_n * g_0$ fails to have total differential of any order at almost all the points of R^n.

These two results complete the ones in [5], [6], [7]. We are indebted to the referee for a simplification of the proof of Theorem A.

1. Notation and definitions

As usual α will denote the n-tuple of integers $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

$$\alpha! = \alpha_1!\, \alpha_2! \ldots \alpha_n!, \quad D^0 f = f$$

The Taylor's expansion of order m will be written as

$$f(x) = \sum_{|\alpha| \le m} \frac{D^\alpha f(z)}{\alpha!}(x - z)^\alpha + \sum_{|\alpha| = m+1} \frac{(m + 1)}{\alpha!}(x - z)^\alpha \int_0^1 (1 - t)^m D^\alpha f(z + t(x - z))\, dt$$

Received July 28, 1975.