NORM-CONSTANT ANALYTIC FUNCTIONS AND EQUIVALENT NORMS

BY

J. GLOBEVNIK[1]

Let X be a complex Banach space, Δ the open unit disc in C and let $f: \Delta \to X$ be an analytic function satisfying $\|f(\zeta)\| \equiv 1$ ($\zeta \in \Delta$). If X is strictly c-convex [1] then by a result of Thorp and Whitley [7] f is a constant (see also [5]). If X is not strictly c-convex then there are always nonconstant analytic functions from Δ to X having constant norm on Δ. Such functions were studied in [2], [3] and certain necessary and sufficient conditions were obtained for an analytic function to have constant norm.

Suppose that a nonconstant analytic function $f: \Delta \to X$ has constant norm on an open subset of Δ. An easy application of the Hahn-Banach theorem shows that such an f does not have any zeros on Δ. This shows that there are many analytic functions from Δ to X whose norm is not constant on any open subset of Δ and in any norm on X, equivalent to the original one. In the present paper we give a surprisingly simple complete description of such functions.

Throughout, Δ is the open unit disc in C. If X is a complex Banach space we denote by $S(X)$, X', $L(X)$ the unit sphere of X, the dual space of X and the Banach algebra of all bounded linear operators from X to X, respectively. The image of $x \in X$ under $u \in X'$ is denoted by $\langle x \mid u \rangle$. If T is a subset of X we denote by $\overline{\text{sp}}\ T$ the closed linear subspace spanned by the elements of T.

THEOREM. *Let X be a complex Banach space and let*

$$f(\zeta) = a_0 + \zeta a_1 + \zeta^2 a_2 + \cdots$$

be a nonconstant analytic function from Δ to X. Then

$$a_0 \notin \overline{\text{sp}}\ \{a_1, a_2, a_3, \ldots\}$$

if and only if there exist an equivalent norm $|||\ \ |||$ on X and an open subset $U \subset \Delta$ such that $|||f(\zeta)|||$ is constant on U.

LEMMA 1. *Let X be a complex Banach space and let $f: \Delta \to X$ be an analytic function. Suppose that $\|f(\zeta)\| \equiv c > 0$ on some open subset of Δ. Then $f(\Delta) \subset f(\zeta_0) + \text{Ker}\ u$ where $\zeta_0 \in \Delta$, $u \in X'$ and $f(\zeta_0) \notin \text{Ker}\ u$.*

Proof. Assume that $\|f(\zeta)\| \equiv c > 0$ ($\zeta \in U$) where $U \subset \Delta$ is an open set and let $\zeta_0 \in U$. By the Hahn-Banach theorem there exists $u \in S(X')$ satisfying $\langle f(\zeta_0) \mid u \rangle = c$. Since $|\langle f(\zeta) \mid u \rangle| \leq \|f(\zeta)\| \cdot \|u\| = c$ ($\zeta \in U$) it follows that

Received July 1, 1975.

[1] This work was supported in part by the Boris Kidrič Fund, Ljubljana, Yugoslavia.