A FACTORIZATION THEOREM FOR COMPACT OPERATORS

BY

GREGORY F. BACHELIS

1. Notation and definitions

If X and Y are Banach spaces, let $K(Y, X)$ denote the compact operators from Y to X with the operator norm, let $F_0(Y, X)$ denote the bounded operators from Y to X with finite-dimensional range, and let $F(Y, X)$ denote the closure of $F_0(Y, X)$ in $K(Y, X)$. If $X = Y$, we write simply $K(X)$, etc.

A Banach space X has the *approximation property* if the identity operator on X can be approximated uniformly on compact subsets of X by operators in $F_0(X)$. If these operators can be taken to have norm less than or equal λ, then X has the *λ-metric approximation property*. Finally, X has the *bounded approximation property* if it has the λ-metric approximation property for some λ.

By "subspace" we mean "closed subspace," and by "isomorphic" we mean "linearly homeomorphic".

2. Statement of results

A theorem of Grothendieck [5, Proposition 35] states that X has the approximation property if and only if $F(Y, X) = K(Y, X)$ for all Banach spaces Y. If $F(Y, X) \neq K(Y, X)$ and $Z = X \oplus Y$, then one easily shows that $F(Z) \neq K(Z)$. However, it is an open question whether $F(X) = K(X)$ implies X has the approximation property. In this paper we prove the following:

THEOREM 1. *If E is a Banach space with the bounded approximation property, and E has a subspace X which fails the approximation property, then E has a subspace Y such that $F(Y, X) \neq K(Y, X)$.*

If in addition E is isomorphic to $E \oplus E$, then E has a subspace S such that $F(S) \neq K(S)$.

For examples of Banach spaces failing the approximation property, the reader is referred to [1], [3], and [7].

The above theorem generalizes a result of Freda Alexander [1]. The author would like to thank Dr. Alexander for making a preprint of [1] available to him.

We note that, if E and X are as above, then the Y produced by the proof of Grothendieck's theorem for which $F(Y, X) \neq K(Y, X)$ is not *a priori* isomorphic to a subspace of E.

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