COMPACT EXTREMAL OPERATORS

BY

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1. Introduction

For X a Banach space, let $\mathscr{B}(X)$ denote the space of bounded linear operators and $\mathscr{C}(X)$ the space of compact linear operators. The identity of a Banach algebra is always an extreme point of its unit ball. See [1]. As a simple consequence, any unitary element is also extreme. Kadison [4] has shown that for X Hilbert space, the extreme points of the unit ball of $\mathscr{B}(X)$ are precisely the semiunitary operators (partial isometries such that either $TT^* = I$ or $T^*T = I$).

For X an arbitrary infinite dimensional Banach space, there is no reason to suspect that $\mathscr{C}(X)$ has many, if indeed any, extreme points in its unit ball. In the first place, $\mathscr{C}(X)$ does not contain any unitary operators. Moreover, the Krein-Millman theorem cannot be readily invoked to conjure up extreme points, since there are no known examples where $\mathscr{C}(X)$ is a conjugate space, and many examples where $\mathscr{C}(X)$ is known not to be a conjugate space. See [2]. Finally, it is known that, for X either Hilbert space or c_0, $\mathscr{C}(X)$ has no extreme points. See [5] for Hilbert space.

We present two results in this paper. First, we show that the unit ball of $\mathscr{C}(l^p)$ is the norm closed convex hull of its extreme points for $1 \leq p < \infty$ and $p \neq 2$. We do so by constructing extreme points which, like unitary operators use all the coordinates. For the bizarre James' space we construct very different extremal operators, not at all analogous to unitary operators.

2. l^p spaces

LEMMA 2.1. *Let $\{e_i\}$ be the standard basis for l^p with $2 < p < \infty$. Suppose $Te_j = \sum_{i=1}^{\infty} a_i e_i$, with each $a_i \neq 0$ and $\|Te_j\| = 1$; and that Te_k is nonzero for some $k \neq j$. Then $\|T\| > 1$.*

Proof. Without loss of generality, we can assume that each $a_i > 0$, since $\|T\| = \|VT\|$ where $Ve_i = (\text{sign } a_i)e_i$. Suppose $Te_k = \sum b_i e_i$. Note that $\|e_j \pm \lambda e_k\|^p = 1 + |\lambda|^p$. We will show that for λ sufficiently small, either $\sum |a_i + \lambda b_i|^p$ or $\sum |a_i - \lambda b_i|^p$ is greater than $1 + |\lambda|^p$.

Suppose $0 < |\lambda b| < a$. By applying Taylor's theorem to

$$f(\lambda) = |a + \lambda b|^p + |a - \lambda b|^p$$

we have

$$|a + \lambda b|^p + |a - \lambda b|^p \geq 2a^p + \lambda^2 \tfrac{1}{2}(p(p-1))b^2[|a + \theta\lambda b|^{p-2} + |a - \theta\lambda b|^{p-2}]$$

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