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KOLMOGOROV'S LAW OF THE ITERATED LOGARITHM FOR BANACH SPACE VALUED RANDOM VARIABLES[1]

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1. Introduction

Let B denote a real separable Banach space with norm $\|\cdot\|$, and throughout $X_1, X_2, \ldots$ are independent B-valued random variables such that $EX_k = 0$ and $E\|X_k\|^2 < \infty$ $(k \geq 1)$. As usual $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ and we write Lx to denote $\log x$ for $x \geq e$ and 1 otherwise. The function $L(Lx)$ is written LLx, and B^* denotes the topological dual of B.

In this paper we establish Kolmogorov's version of the *LIL* [6] for B-valued random variables, and this result will have several corollaries dealing with the *LIL* for i.i.d. sequences. In particular, the recent interesting result of G. Pisier [10, Thèoréme 4.3] will be obtained as an easy corollary (see Corollary 4.1).

To motivate Theorem 3.2 we now turn to the *LIL* for i.i.d. sequences in the Banach space setting, but first we need a bit of terminology.

If (M, d) is a metric space and $A \subseteq M$, $x \in M$, we define the distance from x to A by $d(x, A) = \inf_{y \in A} d(x, y)$. If $\{x_n\}$ is a sequence of points in M, then $C(\{x_n\})$ denotes the cluster set of $\{x_n\}$. That is, $C(\{x_n\})$ is all possible limit points of the sequence $\{x_n\}$. We also will use the notation $\{x_n\} \twoheadrightarrow A$ if both $\lim_n d(x_n, A) = 0$ and $C(\{x_n\}) = A$.

Now let $X_1, X_2, \ldots$ be i.i.d. B-valued random variables such that $EX_1 = 0$ and $E\|X_1\|^2 < \infty$. In view of Strassen's formulation of the Hartman-Wintner result [12] and the recent results in [7], [9], [10] we say *X satisfies the* LIL *in B* if for $X_1, X_2, \ldots$ independent copies of X we have a bounded limit set K in B such that

$$P\{\{S_n/a_n: n \geq 1\} \twoheadrightarrow K\} = 1 \tag{1.1}$$

where $a_n = \sqrt{2nLLn}$ $(n \geq 1)$.

However, (1.1) is not always true under the classical moment assumptions in the infinite dimensional setting, but necessary and sufficient conditions for (1.1) to hold are known, and another will be established in Theorem 4.1 below.

If $\mu = \mathscr{L}(X_1)$ denotes the distribution of X_1, the limit set K turns out to be the unit ball of a Hilbert space H_μ which is uniquely determined by the covariance function

$$T(f, g) = \int_B f(x)g(x)\, d\mu(x) = E(f(X_1)g(X_1)) \qquad (f, g \in B^*). \tag{1.2}$$

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