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ON A PROBLEM OF STOLZENBERG IN POLYNOMIAL CONVEXITY

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1. Introduction

The following problem has been posed by G. Stolzenberg [3, p. 350, Problem 9]: Let $\mathscr{A}$ be a uniform algebra on the unit circle T which is generated by a finite number of functions of constant unit modulus. Show that $\sigma(\mathscr{A})\backslash T$ has the structure of a (possibly empty) one-dimensional analytic space, where $\sigma(\mathscr{A})$ is the spectrum of the Banach algebra $\mathscr{A}$. By using the finite set of n, say, unimodular generators to imbed T into the torus T^n, one can reformulate the problem in a more geometric setting as that of showing, for a Jordan curve Γ contained in T^n, that $\hat{\Gamma}\backslash\Gamma$ is a (possibly empty) one-dimensional analytic subset of $\mathbf{C}^n\backslash\Gamma$, where $^\wedge$ denotes the polynomially convex hull. Our first result includes a solution to this problem as a special case.

We will say that a compact subset Z of T^n is an AC set (a union of a set of Arcs with a polynomially Convex set) provided that there is a compact polynomially convex set $K \subseteq Z$ such that $Z\backslash K$ has the structure of an arc at each of its points. By the latter we mean that for each $p \in Z\backslash K$ there exists a homeomorphism of a neighborhood of p in $Z\backslash K$ with some open interval on the real axis.

THEOREM 1. *Let X be a compact subset of T^n which is contained in an AC set Z. Then $\hat{X}\backslash X$ is a (possibly empty) analytic subset of pure dimension one in $\mathbf{C}^n\backslash X$. Moreover, $\hat{X}\backslash X$ is algebraic, in the sense that there exists a global algebraic subvariety B of $\mathbf{C}^n$ such that $\hat{X}\backslash X = B \cap (\overline{U^n}\backslash T^n)$, and $(\hat{X}\backslash X)^- \cap T^n$ is a union of real analytic curves contained in X.*

Results of this type were first obtained by Wermer [10], Bishop [4], and Stolzenberg [8], under smoothness restrictions, for real curves. In Theorem 1, no smoothness is assumed, but rather there is the geometric hypothesis that X lies in the torus. It is interesting to note, however, that the boundary curves of the hull are shown, a posteriori, by the application of a reflection principle, to be in fact real analytic.

For X a Jordan curve, we can, for the theorem, take $Z = X$ with K empty. When X is a Jordan arc, take $Z = X$ with K the set of two endpoints, to conclude that $\hat{X}\backslash X$ is either analytic or empty. From the argument principle (cf. [8], [11]), it follows that the latter must be the case; i.e., X is polynomially convex. Thus we recover the following result of Stolzenberg [7].

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