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FORMAL AND COFORMAL SPACES

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1. Introduction

In this paper, we apply minimal algebras and minimal Lie algebras to the study of rational homotopy theory. Sullivan has introduced the concept of a formal space [2]. A space is formal if its rational homotopy type is a formal consequence of its cohomology algebra. We give an equivalent definition in terms of perturbations in the differential of a minimal Lie algebra model. These perturbations are related to Massey products, but they have several advantages. For example, perturbations are always defined and they are well defined once we have chosen generators for our model. Take a minimal rational CW complex. Roughly speaking, a perturbation is the deviation that the attaching maps for cells have from being quadratic.

We also introduce a concept which is dual to formality. A space is coformal if its rational homotopy type is a formal consequence of its homotopy Lie algebra. Equivalently, a space is coformal if the k invariants in its rational Postnikov system are quadratic.

The main theorems in this paper are:

COROLLARY 5.2. *Let k be a field of characteristic zero. Two simply connected finite complexes have the same rational homotopy type if there is a k homotopy equivalence such that the induced cohomology isomorphism is rational.*

PROPOSITION 4.4. *Every n connected compact m-dimensional manifold with cohomology of rank > 3 and $m \leq 3n + 1$, $n \geq 1$, is both formal and coformal.*

PROPOSITION 4.6. *Every simply connected compact manifold of dimension ≤ 6 is formal.*

As a corollary of 5.2, we get a result announced by Sullivan [14] and Halperin-Stasheff [16]. A simply connected finite complex is formal over the rationals if it is formal over an extension field. Deligne, Griffiths, Morgan, and Sullivan [2] have shown that compact Kähler manifolds are formal over the reals. Hence, compact simply connected Kähler manifolds are formal over the rationals. (The restriction to simply connected spaces is actually unnecessary [16].)

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