ILLINOIS JOURNAL OF MATHEMATICS
Volume 23, Number 2, June 1979

THE RATIONAL HOMOTOPY GROUPS OF COMPLETE INTERSECTIONS

BY

JOSEPH NEISENDORFER

Introduction

A complete intersection of complex dimension n is a nonsingular subvariety of CP^{n+r} which is the transverse intersection of exactly r nonsingular hypersurfaces. In this paper we compute the rational homotopy groups of all complete intersections of complex dimension greater than one. Formality and the structure of the rational cohomology ring make this computation possible. In fact, our computation is valid for any formal space whose rational cohomology ring looks like that of a complete intersection.

Any nonsingular projective algebraic variety is a compact Kähler manifold. If it is also a complete intersection of complex dimension greater than one, then it is simply connected. By Deligne, Griffiths, Morgan, and Sullivan [2], all the rational homotopy invariants of a simply connected compact Kähler manifold are a formal consequence of the rational cohomology ring. Such a space is called formal. (Actually, [2] shows only that the real homotopy invariants are a formal consequence of the real cohomology ring, but real formality implies rational formality [3], [6], [12].) Equivalently, the rational homotopy invariants of a formal space are a formal consequence of the rational homology coalgebra. Theorem 2 below is a precise formulation of this principle for the rational homotopy groups.

The rational cohomology ring of a complete intersection is not too complicated. Except for powers of the Kähler form, the rational cohomology ring is connected up to the middle dimension [Hirzebruch, 4, Theorem 22.1.2]. Poincaré duality implies that the cup product makes the middle dimensional cohomology group into a nondegenerate bilinear form.

Let V_n be a complete intersection of complex dimension n. The rational homotopy groups $\pi(V_n) \otimes Q$ are complicated enough so that some algebraic structure is needed to describe them. This is given by the Samelson product [13]. More precisely, $\pi_k(V_n) \otimes Q$ is isomorphic to $\pi_{k-1}(\Omega V_n) \otimes Q$ and the Samelson product gives $\pi(\Omega V_n) \otimes Q$ the structure of a graded Lie algebra.

If n is greater than one, V_n has the same rational homotopy type as $X \cup_\alpha e^{2n}$ where X is a bouquet of a single copy of CP^{n-1} and copies of S^n and where $\alpha: S^{2n-1} \to X$ is the attaching map for the top cell e^{2n}. Let h_0 be the number of copies of S^n which occur in X. If h_0 is nonzero, then Theorem 1 below may be expressed as follows: The rational homotopy Lie algebra of ΩV_n is the rational homotopy Lie algebra of ΩX modulo the ideal generated by the homotopy

Received June 29, 1977.