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ON QUASISIMILARITY FOR SUBNORMAL OPERATORS

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In this paper it is shown that if two subnormal operators on separable Hilbert spaces are quasisimilar, then the weak* (or, ultraweakly) closed algebras they generate are isomorphic, and this isomorphism has additional properties. The pure and normal parts of quasisimilar subnormal operators are also investigated, and it is shown that the normal parts must be unitarily equivalent. It is also proved that the pure parts of quasisimilar cyclic subnormal operators must be quasisimilar. These results are then applied to characterize the normal operators all of whose simple parts are quasisimilar. Another application is made to obtain part of a result of W. S. Clary characterizing those subnormal operators that are quasisimilar to the unilateral shift of multiplicity one.

In this paper all Hilbert spaces are separable and all operators are bounded and linear. An operator S on a Hilbert space $\mathscr{H}$ is *subnormal* if there is a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a normal operator N on $\mathscr{K}$ such that $N\mathscr{H} \subseteq \mathscr{H}$ and $S = N \mid \mathscr{H}$ (the restriction of N to $\mathscr{H}$). The weak* topology on $\mathscr{B}(\mathscr{H})$ is the topology $\mathscr{B}(\mathscr{H})$ has as the Banach space dual of $\mathscr{B}_1(\mathscr{H})$, the trace class operators [17]. It is customary to call this the ultraweak topology. The term "weak*" not only obviates the misleading term "ultraweak", but also emphasizes that all the results concerning the dual of a separable Banach space are applicable to $\mathscr{B}(\mathscr{H})$ with its weak* topology.

For S in $\mathscr{B}(\mathscr{H})$, $\mathscr{A}(S)$ denotes the weak* closed algebra generated by S and the identity, 1. That is $\mathscr{A}(S)$ is the weak* closure of $\{p(S)\colon p \text{ is a polynomial}\}$. It has recently been shown by Olin and Thomson [13] that, for a subnormal operator S, $\mathscr{A}(S)$ equals the closure of $\{p(S)\colon p \text{ is a polynomial}\}$ in the weak operator topology (WOT).

If $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces, an operator $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ is said to be quasi-invertible if it is injective and has dense range; that is, if $\ker X = (0)$ and $(\operatorname{ran} X)^- = \mathscr{H}_2$. If $S_j \in \mathscr{B}(\mathscr{H}_j)$ $(j = 1, 2)$, then S_1 is *quasisimilar* to S_2 if there are quasi-invertible operators $X_{21}\colon \mathscr{H}_1 \to \mathscr{H}_2$ and $X_{12}\colon \mathscr{H}_2 \to \mathscr{H}_1$ such that $X_{21}S_1 = S_2X_{21}$ and $X_{12}S_2 = S_1X_{12}$. Denote this by $S_1 \sim S_2$. This equivalence relation of quasisimilarity was introduced by Sz.-Nagy and Foias (see

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