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A SYSTEM OF GAPS IN THE EXPONENT SET OF PRIMITIVE MATRICES

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1. Introduction

A matrix is *nonnegative* (*positive*) if all its entries are nonnegative (positive). A nonnegative square matrix A is *primitive* if $A^k > 0$ for some positive integer k. The smallest such k for the given matrix A is $\gamma(A)$, the *exponent* (of primitivity) of A. Since 1950 when Wielandt [10] first proclaimed the exact general upper bound for γ, there has been a considerable number of papers establishing bounds for special families of primitive matrices. The interested reader is referred to [1], [2], [3], [4], [6], all of which use graph theory as a major tool in the search for γ.

In [2] Dulmage and Mendelsohn reveal what they refer to as *gaps* in the exponent set of primitive matrices. Each gap is a set S of consecutive integers below Wielandt's general bound $W_n = n^2 - 2n + 2$, such that no n-square primitive matrix has an exponent in S. The gaps displayed are

$$n^2 - 3n + 4 < \gamma < (n-1)^2 \quad \text{and} \quad n^2 - 4n + 6 < \gamma < n^2 - 3n + 2.$$

For even n a gap contains the union of the two gaps just mentioned: $n^2 - 4n + 6 < \gamma < (n-1)^2$.

It is the purpose of this paper to disclose a system of such gaps containing the two general gaps just mentioned as special cases. We show that for any integral n and t there is no primitive matrix A of order n for which

$$n^2 - tn + \tfrac{1}{4}(t+1)^2 < \gamma(A) < n^2 - (t-1)n + t - 2.$$

For $t = 3, 4$ these are the gaps shown in [2]. For even n an additional gap is supplied indicating how further gaps may be obtained.

2. Definitions and notations

Let $G(A)$ be the directed graph defined by the nonnegative matrix A. A graph is *primitive* with exponent γ, if it is a graph of a primitive matrix with exponent γ. Let $L(G)$ denote the set of lengths of the simple circuits of G and let $\lambda(G)$ denote the number of the distinct lengths.

It is well known that G is primitive if and only if it is strongly connected and

$$\text{g.c.d. } \{c \mid c \in L(G)\} = 1.$$

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