ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 1, Spring 1981

INVARIANCE PROPERTIES OF FINITELY ADDITIVE MEASURES IN R^n

BY
STANLEY WAGON

0. Introduction

After the existence of a non-Lebesgue measurable set had been proved, the problem of the existence of a total (i.e., defined for all sets), finitely additive, congruence invariant measure (with values in $[0, \infty]$) in $\mathbf{R}^n$ was considered. By congruence (or isometry) of $\mathbf{R}^n$ we mean a distance-preserving bijection from $\mathbf{R}^n$ to $\mathbf{R}^n$; in $\mathbf{R}^1$ there are only translations and reflections about a fixed point. In 1914, Hausdorff [9, p. 469] (see also [21, pp. 74 and 97]) constructed a paradoxical decomposition of the 2-sphere (a construction central to the more well-known Banach-Tarski paradox [3] (see also [24])) which implies that no such measure exists in $\mathbf{R}^n$ if $n \geq 3$. And in 1923, Banach [2] proved that such measures do indeed exist in $\mathbf{R}$ and $\mathbf{R}^2$, and can be chosen to extend Lebesgue measure. To prove this for $\mathbf{R}$, Banach developed the ideas of the Hahn-Banach Theorem to extend Lebesgue measure to a total, finitely additive, translation invariant measure ν. Then it suffices to define $\mu(A) = \frac{1}{2}(\nu(A) + \nu(-A))$. (See [19, p. 193] or [12, p. 359] for details.) The measure in $\mathbf{R}^2$ was obtained by applying a clever integration technique to the measure in $\mathbf{R}$.

The modern approach to these classical theorems of Banach uses the notion of an amenable group, invented by von Neumann [17]. In that paper von Neumann realized that Banach's techniques generalize to any group bearing an appropriate measure; such groups are called amenable. This yielded total measures in higher dimensions: if G is an amenable group of congruences of $\mathbf{R}^n$, then a total, finitely additive, G-invariant extension of Lebesgue measure exists.

Banach's theorem for $\mathbf{R}$ raises the following question, which motivated the work of this paper. Can a total, finitely additive extension of Lebesgue measure be invariant under translations, but not reflections? More generally, letting Inv (μ), for μ a total finitely additive extension of Lebesgue measure in $\mathbf{R}^n$, be the group of congruences with respect to which μ is invariant, we have the question of which groups arise as Inv (μ), for some μ. The following theorem, which is proved in Section 3, gives a necessary condition for a group to be realized in this way, which is applicable to most of the interesting cases (in particular, it follows that all groups of congruences of $\mathbf{R}$ or $\mathbf{R}^2$ are realizable).

THEOREM 1. *If G is an amenable group of congruences of $\mathbf{R}^n$ then there is a*

Received March 5, 1979.