ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 2, Summer 1981

WEIGHTED KERNEL FUNCTIONS AND CONFORMAL MAPPINGS

BY

WILLIAM S. COHN

Introduction

Let D be a domain in the plane bounded by $n+1$ analytic Jordan curves. Garabedian [5] and Nehari [6] consider the following extremal problem. Suppose h is positive and continuous on ∂D. For $\zeta \in D$ let $S = \{f, f \text{ holomorphic and bounded on } D, f(\zeta) = 0, \text{ and } |f| < h \text{ on } \partial D\}$. What is $\sup_{f \in S} |f'(\zeta)|$?

Within the framework of this problem certain functions arise naturally. These are the "reproducing kernels" $B(z, \zeta, h^2)$, holomorphic in $z \in D$ which satisfy

$$f(\zeta) = \int_{\partial D} f(\eta)\overline{B(\eta, \zeta, h^2)}h^2 |d\eta|$$

for f holomorphic on $\bar{D}$, the closure of D.

It is the purpose of this paper to study these kernels from the point of view of the Hardy class, $H^2(D)$. The basic technique is to make simple changes in h^2 and calculate the resulting change in $B(z, \zeta, h^2)$. This amounts to varying the inner product on $H^2(D)$.

Our main results are Theorem 5.2 and 5.4. Theorem 5.4 may be regarded as a generalization of the identity

$$\text{(1)} \qquad \frac{2(1-\bar{\zeta}z)}{(1-\bar{\zeta}e^{i\theta})(1-ze^{-i\theta})} = \frac{e^{i\theta}+z}{e^{i\theta}-z} + \frac{e^{-i\theta}+\bar{\zeta}}{e^{-i\theta}-\bar{\zeta}}$$

which holds for $|\zeta| < 1$, $|z| < 1$.

This identity expresses a relationship between the H^2 reproducing kernel and the kernel

$$\frac{e^{i\theta}+z}{e^{i\theta}-z}$$

used in the integral representation of a singular inner function defined on the unit disk. We recall that

$$s(z) = \exp\left(-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\sigma(\theta)\right)$$

Received May 24, 1979.