ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 2, Summer 1981

ON DIFFERENTIATION OF MULTIPLE TRIGONOMETRIC SERIES

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1. In this paper we obtain some multidimensional analogues to the theorems of Riemann and Lebesgue on the differentiation of formally integrated one dimensional trigonometric series. To develop these results we define symmetric derivatives for functions of several variables by expanding weighted spherical means of the functions into power series of even or odd terms. We use surface harmonics for the weights. We show that for each surface harmonic a weighted symmetric derivative can be defined, and that for each weighted symmetric derivative a different theorem of "Riemann type" can be constructed.

In p dimensions, $p \geq 2$, we let

$$x = (x_1, \ldots, x_p) \in E_p \quad \text{and} \quad n = (n_1, \ldots, n_p) \in \mathbf{Z}_p.$$

We let $\Sigma = \{x \in E_p\colon |x| = 1\}$ and $x' = |x|^{-1}x$. We write $ds(\eta)$ to denote the surface element in $(p-1)$-dimensional surface integrals. Let ν be a non-negative integer and let $S_\nu(x)$ be a harmonic polynomial homogeneous of degree ν. For $\xi \in \Sigma$, let $\Omega(\xi) = S_\nu(\xi)$. Ω is called a *surface harmonic* of degree ν.

Suppose a function $F(x)$ is defined in a neighborhood of $x_0 \in E_p$ and is integrable over the surface of each sphere $|x - x_0| = t$, for t small. Let k be an integer of the form $k = \nu + 2r$, where r is a non-negative integer. We make the following definition.

DEFINITION. F has at x_0 a kth Ω-derivative with value s_k if

$$(1.1)\qquad (2\pi)^{-p/2}\int_{\eta \in \Sigma} F(x_0 + t\eta)\Omega(\eta)\, ds(\eta) = a_\nu t^\nu + a_{\nu+2}t^{\nu+2} + \cdots + a_k t^k + o(t^k),$$

as $t \to 0$, where

$$a_k = \frac{2^{-p/2-k+1}s_k}{((k-\nu)/2)!\,\Gamma((k+\nu+p)/2)}.$$

This definition may be thought of as an analogue to the definitions (1.2) and (1.3) of [10, volume 2], p. 59, depending on whether ν is even or odd.

When $p = 2$, $\nu = 0$, $k = 2r$, and $\Omega(\xi) \equiv 1$, our formula (1.1) gives the rth *generalized Laplacian*, which is developed in [9]. When $p = 2$, $\nu = 1$, and

Received April 17, 1979.