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DISCRETENESS CRITERIA AND HIGH ORDER GENERATORS FOR SUBGROUPS OF $SL(2, R)$

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1. Introduction

In 1939 and 1940 Lauritzen [3] and Nielson [11] used noneuclidean geometry to prove:

(*) A nonabelian group in $SL(2, R)$ which contains at most hyperbolics and $\pm I$ must be discrete.

Later in 1940 Siegel found a striking proof [12] of the more general theorem:

(**) If the matrices in a group Γ in $SL(2, R)$ do not all have a real (or infinite) fixed point in common, and if they do not all map a pair of real points onto that very pair, then Γ is discrete provided Γ contains no sequence of elliptics converging to I.

Authors presenting these or similar ideas now all use Siegel's proof, but some [10] present (**) while others [6], [7], [9] prefer the more easily grasped but weaker (*).

A formulation permitting a proof somewhat shorter than those for (*) and (**) is found by casting Siegel's proof, with minor modifications, upon class (i) of Theorem 1 of this paper. The result is Theorem 5 below which applies to a slightly wider collection of subgroups, and surprisingly also has more explicit conclusions; (*) and (**) are direct consequences of this theorem and its corollary respectively. The few subgroups for which this theorem does not apply are of a particularly simple nature, and are easily analyzed (classes (ii), (iii), and (iv) of Theorem 1).

The classification of subgroups into the four classes mentioned above also allows us to prove the existence of high order generators for subgroups of $SL(2, R)$ (Theorems 2 and 4).

In this and the next two sections all matrices are assumed to be in the Special Linear group $SL(2, R)$ of two by two matrices with real entries and determinants one.

If

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

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