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GROUPS OF THE SECOND KIND WITHIN THE MODULAR GROUP

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The modular group Γ can be studied using (hyperbolic) geometry, group theory or graph theory. The division of the subgroups into those of the first and of the second kind has always been described in geometrical terms. We provide other criteria. We begin by discussing the various views of the whole group.

Geometrically, Γ is $LF(2, \mathbf{Z})$, the group of integral bilinear transformations of the Poincaré plane

$$\mathscr{H} = \{z \in \mathbf{C}: \mathrm{Im}(z) \geq 0\} \cup \{\infty\}.$$

Is is generated by $x = \pm(0 \ \ -1 \,|\, 1 \ \ 0)$ and $y = \pm(0 \ \ -1 \,|\, 1 \ \ 1)$. In fact,

$$\Gamma = \langle x, y: x^2 = y^3 = I \rangle, \tag{1}$$

so that each element of Γ can be written uniquely as a word in x, y and y^2. The element $u = \pm(1 \ \ 1 \,|\, 0 \ \ 1)$ is very significant. The elements x, y and u generate, respectively, the Γ-stabilizers of i, $\phi = \frac{1}{2}(-1 + i\sqrt{3})$ and ∞.

Let G be a subgroup of index n in Γ; we allow $n = \infty$. The Γ-orbits of i, ϕ and ∞ each split into G-orbits. The G-stabilizers of points in the same G-orbit are clearly G-conjugate. Let r (resp. s, h_0) be the number of G-orbits in Γi (resp. $\Gamma\phi$, $\Gamma\infty$) which have non-trivial G-stabilizers. Let h_∞ be the number of other G-orbits in $\Gamma\infty$. These are the geometrical definitions of the parameters which appear in [6].

As a group, Γ is the free product of cyclic groups of order two and three, see (1). By a careful application of the ideas behind the Kurosh subgroup theorem, the subgroup G has a presentation

$$\langle x(1), \ldots, x(r), y(1), \ldots, y(s), u(1), \ldots, u(h_0), a(1), \ldots, a(t_1): \\ x(1)^2 = \cdots = x(r)^2 = y(1)^3 = \cdots = y(s)^3 = w = I \rangle, \tag{2}$$

where w is a known word in the generators, and is trivial when $n = \infty$. In (2), each $x(j)$ (resp. $y(j)$) is a Γ-conjugate of x (resp. y), and each $u(j)$ a Γ-conjugate of $u^{m(j)}$ for some non-zero integer $m(j)$. The parameter t_1 is equal to twice the genus of $\mathscr{H}/G$ when n is finite. The existence of the presentation (2) is proved group-theoretically in [2], [3]. For n finite, it is proved geometrically in [4], for n infinite it is proved graph-theoretically in [6]. The parameters r, s, and h_0

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