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THE VARIETY OF POINTS WHICH ARE NOT SEMI-STABLE

BY

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1. Introduction

(1.1) **Background.** Let k be an algebraically closed field and let V be a finite-dimensional vector space over k. Let G be a reductive algebraic subgroup of $GL(V)$.

Let $k[V]$ be the algebra of regular functions on V. The group G acts on $k[V]$ as follows:

$$(g \cdot f)(v) = f(g^{-1}v)$$

for all $f \in k[V]$, $g \in G$, and $v \in V$. The ring of G-invariant functions on V is

$$k[V]^G = \{f \in k[V]: g \cdot f = f \quad \text{for all } g \in G\}.$$

We define an algebraic subvariety X of V by

$$X = \{v \in V: f(v) = 0 \text{ for each non-constant homogeneous } f \in k[V]^G\}.$$

A point in V, not in X, is called *semi-stable*.

In order to describe the points in X, it is useful to introduce the concept of an orbit. Let v be in V. The *orbit* of v with respect to the action of G is

$$G \cdot v = \{g \cdot v: g \in G\}$$

The Zariski-closure of $G \cdot v$ will be denoted by cl $(G \cdot v)$.

THEOREM. *Let G be a connected reductive algebraic subgroup of $GL(V)$. Let $v \in V$. The following statements are equivalent:*

(a) *v is not semi-stable*;

(b) $0 \in \text{cl}\,(G \cdot v)$;

(c) *there is a one-parameter subgroup λ of G so that $\lambda(\alpha) \cdot v \to 0$ as $\alpha \to 0$.*

The notation in (c) will be explained in (2.1). The equivalence of (a), (b), and (c) is proved in [10; Sections 1 and 2] taking into account [5].

(1.2) **Summary of results.** The purpose of this paper is to prove some results aimed at explicitly describing the set X. The basic theorem is proved in (2.2). As consequences of this theorem, the following corollaries are proved in (2.3) and (3.2).

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