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ON NON-NORMAL SUBGROUPS OF $GL_n(A)$ WHICH ARE NORMALIZED BY ELEMENTARY MATRICES

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1. Introduction

Let A be a ring with identity, $\mathfrak{q}$ a (two-sided) ideal in A (possibly $\mathfrak{q} = A$) and let $f : GL_n(A) \to GL_n(A/\mathfrak{q})$ be the natural homomorphism. We put

$$G = GL_n(A), \quad G(\mathfrak{q}) = \operatorname{Ker} f \quad \text{and} \quad H(\mathfrak{q}) = f^{-1}(C),$$

where C is the centre of $GL_n(A/\mathfrak{q})$. (By definition

$$H(\mathfrak{q}) = \{X \in G : X \equiv xI \ (\mathrm{mod}\ \mathfrak{q}), \text{ where } x\ (\in A) \text{ is central } (\mathrm{mod}\ \mathfrak{q})\}$$

and $H(A) = G(A) = G$.)

Let Δ be the subgroup of G generated by all the elementary matrices $I + aE_{ij}$, $a \in A$, $i \neq j$, $1 \leq i, j \leq n$, and let $\Delta(\mathfrak{q})$ be the *normal* subgroup of Δ generated by the $\mathfrak{q}$-elementary matrices, $I + qE_{ij}$, $q \in \mathfrak{q}$, $i \neq j$. (By definition $\Delta = \Delta(A)$.) Finally, if H, K are subgroups of G, $[H, K]$ is the subgroup generated by commutators $[h, k] = h^{-1}k^{-1}hk$, $h \in H$, $k \in K$.

Our starting point is the following:

THEOREM 1. *Assume that either*

(a) *A satisfies $SR_t(A)$, for some $t \geq 2$,* and $n \geq \max(t, 3)$,

or

(b) *A is finitely generated as a module over its centre and $n \geq 3$.*

If E is a subgroup of G normalized by Δ, then for some unique ideal $\mathfrak{e}$ (called the level of E),

$$\Delta(\mathfrak{e}) \leq E \leq H(\mathfrak{e}).$$

Parts (a) and (b) are due to Bass [1, p. 240] and Vaserstein [11], respectively. Many special cases of this result have appeared over the last twenty years. Among the most important are those due to Brenner [3] ($A = \mathbf{Z}$, $n \geq 3$) and Golubchik [4] (A commutative, $n \geq 3$). The classical example of the modular group shows that the restriction $n \geq 3$ is necessary. It is known [5] that, if N is a normal subgroup of finite index in $GL_2(\mathbf{Z})$, then, with finitely many excep-

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