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CURVATURE AND EULER CHARACTERISTIC FOR SIX-DIMENSIONAL KÄHLER MANIFOLDS

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0. Introduction

Perhaps the most basic problem in Riemannian geometry is the determination of which Riemannian metrics a given manifold can support, in particular which curvature properties can be realized on the manifold. A classical conjecture, due to H. Hopf, is that the Euler characteristic is a basic obstruction to the existence of a metric of nonnegative (or nonpositive) curvature; specifically, if M is a compact, $2n$-dimensional Riemannian manifold, with sectional curvature r,

$$(*)\qquad \begin{aligned} r \geq 0 \quad &\text{implies} \quad \chi(M) \geq 0, \\ r \leq 0 \quad &\text{implies} \quad (-1)^n\chi(M) \geq 0. \end{aligned}$$

This conjecture can be verified in dimensions 2 and 4 by the Gauss–Bonnet–Chern theorem (GBC) [4] (the 4-dimensional result is due to J. Milnor). The purpose of this article is to prove (*) for 6 real-dimensional Kähler manifolds.

The approach taken here is similar to that outlined in [4]. By the GBC, the Euler characteristic $\chi(M)$ is given by the integral over M of a homogeneous polynomial of degree n in the components R_{ijkl} of the Riemann curvature tensor R of M, which we denote by $\chi(R)$. We prove (*) in the case at hand by showing that $\chi(R) \geq 0$ at each point of M.

This theorem is actually a result in what B. O'Neill has dubbed "pointwise geometry", as only algebraic properties of R at a single point of M are used. This pointwise result does not hold in greater generality; that is, there are algebraic curvature tensors R with nonnegative sectional curvature but $\chi(R) < 0$. In [5], R. Geroch has found such an example in dimension six, which is of course non-Kählerian. More recently, Bourguignon and Karcher [3] have found a one-dimensional family of such tensors that are quite nearly Kählerian, the only non-Kähler component being a multiple of the identity operator. These results do not provide a counterexample to (*), however, since to our knowledge no compact manifold has been constructed realizing one of these operators as its curvature tensor at every point.

It should be pointed out that under the stronger hypothesis that the sectional curvature is strictly positive much stronger results have recently been shown. Block and Gieseker [2] have shown that any algebraic vector bundle

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