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THE CLASSIFYING SPACE OF SMOOTH FOLIATIONS

BY

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One of the most striking examples of a foliated manifold was constructed by Thurston [24] on the 3-sphere. He exhibited a family of codimension-one foliations on S^3 for which the Godbillon-Vey class takes on a continuous range of values. In subsequent explicit constructions of foliations with non-zero secondary classes, almost always a discrete group Γ is given which acts on S^q and the foliation is defined on the quotient $M^n = B \times_\Gamma S^q$, where Γ acts freely on B [1], [5], [6], [13], [19], [21]. The quotient space M is very complicated as a manifold, and it is natural to ask whether there exist codimension q foliations on spheres S^n with non-vanishing secondary classes for $q > 1$ (Haefliger's problem 2, p. 241 of [22].) One of the aims of this paper is to answer this question affirmatively (Corollary 4.6). For $n = 2q + 1$, if S^n admits a rank q subbundle $Q \subseteq TS^n$ then there exists a family of codimension q foliations on S^n for which a set of secondary classes takes on a continuous range of values. Other values of $n > 2q$ also work, and S^n can be replaced with any closed, oriented n-manifold M which admits a q-frame field $Q \subseteq TM$. This is a consequence of Thurston's realization theorem [25], and the following two theorems which we will prove.

For each $q > 1$ a sequence of non-negative integers $\{v_{q,n}\}$ is defined in §2.8 with the properties:

(1) For $q = 2$, $\lim_{k\to\infty} v_{2,4k+1} = \infty$ and $v_{2,4k+1} > 0$ for all $k > 0$.
(2) For $q = 3$, $\lim_{k\to\infty} v_{3,3k+1} = \infty$ and $v_{3,3k+1} > 0$ for all $k > 1$.
(3) For $q > 3$, $\lim_{n\to\infty} v_{q,n} = \infty$.

We denote by $B\Gamma_q$ Haefliger's classifying space of codimension q smooth foliations. The *integral* homotopy groups of $B\Gamma_q$ are denoted $\pi_*(B\Gamma_q)$.

THEOREM 1. *For each $q > 1$ and $n > 2q$ there is an epimorphism of abelian groups*

$$\pi_n(B\Gamma_q) \to \mathbf{R}^{v_{q,n}}.$$

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