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FAILURE OF F-PURITY AND F-REGULARITY IN CERTAIN RINGS OF INVARIANTS

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1. Introduction

Let $\mathbb{F}_q$ be a finite field of characteristic p, K a field containing it, and $R = K[X_1, \ldots, X_n]$ a polynomial ring in n variables. The general linear group $GL_n(\mathbb{F}_q)$ has a natural action on R by degree preserving ring automorphisms. L. E. Dickson showed that the subring of elements which are fixed by this group action is a polynomial ring [Di], though for an arbitrary subgroup G of $GL_n(\mathbb{F}_q)$, the structure of the ring of invariants R^G may be rather mysterious. If the order of the group $|G|$ is relatively prime to the characteristic p of the field, there is an R^G-linear retraction ρ: $R \to R^G$, the *Reynolds operator*. This retraction makes R^G a direct summand of R as an R^G-module, and so R^G is F-regular. However when the characteristic p divides $|G|$, this method no longer applies, and the ring of invariants R^G need not even be Cohen-Macaulay. M.-J. Bertin showed that when R is a polynomial ring in four variables and G is the cyclic group with four elements which acts by permuting the variables in cyclic order, then the ring of invariants R^G is a unique factorization domain which is not Cohen-Macaulay, providing the first example of such a ring, [Be]. Related work and bounds on the depth of R^G can be found in the work of R. M. Fossum and P. A. Griffith; see [FG]. More recently D. Glassbrenner studied the invariant subrings of the action of the alternating group A_n on a polynomial ring in n variables over a field of characteristic p, constructing examples of F-pure rings which are not F-regular [G1], [G2]. Both these families of examples study rings of invariants of $K[X_1, \ldots, X_n]$ under the action of a subgroup G of the symmetric group on n elements, i.e., an action which permutes the variables, and Glassbrenner shows that for such a group the ring of invariants is F-pure, see [G1, Proposition 0.6.7].

We shall construct examples which demonstrate that the ring of invariants for the natural action of a subgroup G of $GL_n(\mathbb{F}_q)$ need not be F-pure. We shall obtain such examples with the group G being the symplectic group over a finite field. These non F-pure invariant subrings are always complete intersections, and are actually hypersurfaces in the case of $G = Sp_4(\mathbb{F}_q) < GL_4(\mathbb{F}_q)$ acting on the polynomial ring $R = K[X_1, X_2, X_3, X_4]$. These examples are particularly interesting if one is attempting to interpret the Frobenius closures and tight closures of ideals as contractions from certain extension rings, since we have an ideal generated by a system of

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