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MEROMORPHIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WHICH ARE NOT SOLUTIONS OF FIRST ORDER ALGEBRAIC DIFFERENTIAL EQUATIONS

BY

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1. Introduction

In [11], Siegel proved that no nontrivial solution of Bessel's equation

$$z^2y'' + zy' + (z^2 - \alpha^2)y = 0$$

can satisfy a first order algebraic differential equation (ADE) with coefficients in the field of rational functions, provided that α is not one-half of an odd integer. This result was extended by Bank [1], who proved that no nontrivial solution of the above equation will satisfy a first order ADE with coefficients in the field of meromorphic functions of order less than one, the condition on α remaining the same.

Here we consider equations of the form

$$w'' + Pw' + Qw = 0, \tag{1.1}$$

where P and Q are meromorphic functions of finite order. We assume that all solutions of (1.1) are meromorphic, and find conditions under which solutions do not satisfy first order ADE with suitable coefficients. We require a lemma due to Siegel [11], which we state in a less general form.

DEFINITION. *A differential field L of meromorphic functions is a field of meromorphic functions which contains derivatives of all its elements.*

If α is any nonnegative real number we denote by L_α the field of all meromorphic functions of order less than or equal to α. Clearly L_α is a differential field of meromorphic functions.

Siegel's lemma is as follows:

LEMMA 1. *Let L be a differential field of meromorphic functions and let P and Q in* (1.1) *belong to L. Let the solutions of* (1.1) *be meromorphic. Suppose*

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