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A COBORDISM OBSTRUCTION TO EMBEDDING MANIFOLDS

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1. Introduction

Let M be a smooth, compact, stably almost complex manifold and υ a complex normal bundle for M, of large dimension. Let RP^n denote real projective space of dimension n and ξ the nontrivial real line bundle over RP^n. Then the external tensor product $\upsilon \otimes \xi$ is a bundle over $M \times RP^n$, and it is the real bundle underlying the complex bundle $\upsilon \otimes_C (\xi \otimes C)$; thus $\upsilon \otimes \xi$ is oriented in complex cobordism theory $MU^*(\ )$. We shall prove:

THEOREM 1.1. *If M embeds in euclidean space with codimension $2l$ and the complex dimension of υ is $l + k$, then the Euler class of $\upsilon \otimes \xi$ in $MU^*(M \times RP^{2k})$ vanishes, provided the natural map*

$$MU^{ev}(M) \otimes_{\pi_* MU} MU^*(RP^{2k}) \to MU^{ev}(M \times RP^{2k}) \tag{1.2}$$

is an isomorphism.

It is well known that if M immerses in codimension $2l$ then the Euler class $e(\upsilon \otimes \xi)$ vanishes over $M \times RP^{2k-1}$. For if υ_0 is the normal bundle of an immersion then one has $\upsilon \cong \upsilon_0 \oplus 2k$ as real bundles, and thus

$$\upsilon \otimes \xi \cong \upsilon_0 \otimes \xi \oplus 1 \otimes 2k\xi;$$

this implies that $\upsilon \otimes \xi$ has a nonvanishing section over $M \times RP^{2k-1}$, because $2k\xi$ has a nonvanishing section over RP^{2k-1}. The content of (1.1) is then that, under appropriate hypotheses, if M embeds in codimension $2l$ then $e(\upsilon \otimes \xi)$ must vanish over the larger space $M \times RP^{2k}$. In the case of Euler classes in singular cohomology with Z_2 coefficients this amounts to the fact that the highest Stiefel Whitney class of the normal bundle of an embedding is zero. In Z_2 cohomology one always has the Künneth theorem in its strong form, and all manifolds are oriented; in this light the hypotheses of (1.1) seem reasonable. The map (1.2) is injective by the Künneth theorem for MU theory [6]; the

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