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NILPOTENT ELEMENTS IN GROTHENDIECK RINGS

BY

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This work was inspired by the following theorem of Bass and Guralnick [3]: Let P and Q be finitely generated projective modules over a commutative ring. Then $\otimes^m P \cong \otimes^m Q$ for some $m > 0$ if and only if $\oplus^n P \cong \oplus^n Q$ for some $n > 0$. Their proof depends on the fact [1, Ch. IX, 4.4] that $([P] - [Q])^{d+1} = 0$ in $K_0(R)$, whenever P and Q are finitely generated projective modules with the same rank and R is Noetherian and d-dimensional.

In Section 2 of this paper we show that the nilpotency theorem, suitably interpreted, is valid without the assumption that P and Q be projective. In Section 3 we investigate to what extent the Bass-Guralnick theorem generalizes to non-projective modules, and we obtain some positive results in dimensions one and two. In the fourth section we prove the nilpotency theorem for finitely presented modules over arbitrary (non-Noetherian) rings. The bound d = Krull dimension always suffices, and in at least one case we can do much better: For the ring of continuous real-valued functions on a compact Hausdorff space X one can take d = covering dimension of X.

I want to thank the referee for reading this paper carefully and for suggesting some substantial improvements, particularly in Section 3.

1. Notation and preliminaries

All rings are commutative, and modules are always finitely generated. If M and N are R-modules, we write MN for the tensor product $M \otimes_R N$, and M^r for the r-fold tensor power, with $M^0 = R$. Similarly, $M + N = M \oplus N$ and $rM = \oplus^r M$. If $f \in \mathbf{Z}^+[X_1, \ldots, X_n]$, the set of polynomials with nonnegative integer coefficients, and $M_1, \ldots, M_n$ are R-modules, the expression $f(M_1, \ldots, M_n)$ makes sense. For an arbitrary polynomial $f \in \mathbf{Z}[X_1, \ldots, X_n]$, let f^+ (respectively $-f^-$) be the sum of all the monomials of f with positive (respectively negative) coefficients. Then $f = f^+ - f^-$, and f^+ and f^- are in

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