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ON PERMUTATION REPRESENTATIONS

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In Memory of Irving Reiner

By a *permutation representation* (G, X) we mean a group G together with a left G-set X. The *orbit set* $G \backslash X$ has as its elements the orbits $[x] = Gx$ for $x \in X$. The question which we shall address is to what extent the permutation representation can be recovered from information about orbit sets alone. Its motivation comes from homotopy theory and we shall indicate below how our remarks apply there.

Evidently from $G \backslash X$ alone we cannot reconstruct G and X. But for any set U the *U-th* power of X, i.e., the set X^U of functions x: $U \to X$, is again a G-set, with $(gx)i = g(xi)$ for $g \in G$, $i \in U$. Furthermore if F: $U \to V$ then X^f: $X^V \to X^U$, the composition with f, is a G-equivariant map. Thus

$$U \mapsto G \backslash X^U, f \mapsto G \backslash X^f$$

defines a functor $\mathrm{Orb}(G, X)$: $\mathrm{Sets}^{\mathrm{op}} \to \mathrm{Sets}$, the *orbit-functor* of (G, X). We shall see that from this functor we can indeed reconstruct, up to a suitable equivalence, the permutation representation.

1. Orbital functors

We shall adopt the following conventions. A natural number n is the set $\{0, 1, \ldots, n-1\}$ of its predecessors, so that $0 = \emptyset$. If f: $m \to m'$ and g: $n \to n'$ are maps of natural numbers then $f + g$: $m + n \to m' + n'$ is the ordinal sum in the obvious sense. For any set W, $\underline{W}$: $0 \to W$ and $\overline{W}$: $W \to 1$ are the unique maps; we shall also on occasion write W for the identity map. For sets U, V, Δ: $U \to U^V$ denotes the generalized diagonal map. If F: $\mathrm{Sets}^{\mathrm{op}} \to \mathrm{Sets}$ then the natural transformation dg: $F(U \times V) \to (FU)^V$ is defined by $\mathrm{pr}_j \mathrm{dg} = F(U \times j)$ where j: $1 \to V$ is an element of V.

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