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SWAN MODULES AND ELLIPTIC FUNCTIONS

BY

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Dedicated to the memory of Irving Reiner

1. Introduction

The normal basis theorem for a finite Galois extension N/L states that N as an L-vector space has a basis of the form $\{a^{\gamma}\}$, where a is a fixed element of N and γ runs over the Galois group $\mathrm{Gal}(N/L) = \Gamma$. In other words, N is a free rank one $L\Gamma$-module. The analogous question for the rings of algebraic integers, $\mathcal{O}_N$ in N and $\mathcal{O}_L$ in L, is the well-known normal integral basis problem. In fact, $\mathcal{O}_N$ is an $\mathscr{A}$-module where

$$\mathscr{A} = \{x \in L\Gamma : \mathcal{O}_N x \subseteq \mathcal{O}_N\},$$

the associated order of the extension N/L in $L\Gamma$. Thus the best possible result would be that $\mathcal{O}_N$ is $\mathscr{A}$-free.

Abelian extensions of $\mathbf{Q}$ are contained in cyclotomic extensions. Leopoldt [4] has shown that in the case that $L = \mathbf{Q}$ and Γ is abelian, $\mathcal{O}_N$ is, indeed, $\mathscr{A}$-free. Furthermore, he has described the order $\mathscr{A}$ explicitly. In the relative case, where both L and N are cyclotomic fields we have the following result [1, Chapter I].

PROPOSITION (1.1). *Let m, r be positive integers such that m divides r. Let $N = \mathbf{Q}(\zeta)$, $L = \mathbf{Q}(\zeta^m)$ where ζ is a primitive mr-th root of unity in $\mathbf{C}$. Let $\Gamma = \mathrm{Gal}(N/L)$ and $\mathscr{A}$ be the associated order of N/L in $L\Gamma$. Then, $\mathcal{O}_N$ is a free rank one $\mathscr{A}$-module.*

Abelian extensions of a quadratic imaginary number field are obtained by adjoining singular values of certain elliptic functions. In [10], M.J. Taylor has obtained elliptic analogues of (1.1) for certain Kummer extensions with respect to an elliptic group law. Taylor showed that the ring of algebraic

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