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# ITERATED INTEGRALS AND EPSTEIN ZETA FUNCTIONS WITH HARMONIC RATIONAL FUNCTION COEFFICIENTS

BY

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**In memory of Kuo-Tsai Chen**

## 1. Introduction

Theta functions (of one complex variable $z$ in the upper half plane) with harmonic polynomial coefficients are well known ([4], [5], see (2.1) below). They satisfy a transformation formula (2.2) under $z \to -z^{-1}$ and their Mellin transforms are Epstein zeta functions of $s$ (see (2.3), (2.4)) which satisfy a corresponding functional equation (2.5) under $s \to k - s$ ($k$ a constant). Using Chen's iterated integrals we find in this paper theta functions with certain harmonic *rational function* coefficients which (when a polynomial coefficient theta function is added) satisfy the same transformation formula (but they are not modular forms). Corresponding Epstein zeta functions satisfy the classical functional equation (2.9 below). We study a particular example related to the Fermat quartic $F_4$: $X^4 + Y^4 = 1$ and its Jacobian $J(F_4)$ [1]. Here the value at $s = 1$ of the Epstein zeta function with rational function coefficients divided by the product of the $L$-functions of two elliptic curves (namely $Y^2 = X^3 \pm 4X$) generates the Abel Jacobi image, in $\mathbf{C}/\mathbf{Z}(i)$, of the 1-cycle in $J(F_4)$ given by $[F_4] - [\iota(F_4)]$. (We consider only the Abel-Jacobi image in

$$H^{3,0}(J(F_4))^*/H_3(J(F_4);\mathbf{Z})).$$

## Section 2

We recall now the formulas defining the theta and Epstein zeta functions associated to a real symmetric positive definite $h \times h$ matrix $Q$, two vectors $A, B \in \mathbf{R}^h$, and a (non-zero) homogeneous polynomial $P(X)$ of degree $g$ in $h$

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