ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, Number 2, Summer 1990

ON A GENERALIZATION OF CHEEGER-CHERN-SIMONS CLASSES[1]

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In memory of Kuo-Tsai Chen

0. Introduction

For foliated bundles in the sense of Kamber-Tondeur [K-T2] there are well-known second order characteristic classes with real coefficients. For these the domain of definition is the cohomology of the truncated Weil algebra $W(G, K)_k$ where G is the structure group, $K \subseteq G$ is maximal compact and k is the codimension of the foliation. On the other hand there are also associated Cheeger-Chern-Simons classes with $\mathbf{R}/L$ coefficients ($L \subseteq \mathbf{R}$ is a subring, e.g., $L = \mathbf{Z}$ or $L = \mathbf{Q}$) defined for each invariant polynomial on the Lie algebra $\mathfrak{g}$ of degree greater than k together with a lift to $H^*(BG, L)$ of the corresponding primary characteristic class in $H^*(BG, \mathbf{R})$ (see [Cn-Si] or [Cr-Si]). In the present paper we combine these two ideas and define (in Section 2) a generalized Cheeger-Chern-Simons class corresponding to a class in the cohomology of the truncation ideal $FW(G, K) \subseteq W(G, K)$ together with a lift to $H^*(BK, L)$ of the image in $H^*(W(G, K)) = H^*(BK, \mathbf{R})$. Thus, for instance, a real second order characteristic class corresponds to a Cheeger-Chern-Simons class with $L = 0$ via the coboundary map

$$H^*(W(G, K)_k) \xrightarrow{\delta} H^*(FW(G, K))$$

except that the latter has an indeterminacy consisting of primary classes.

One might hope that this approach for $L \neq 0$ gives other interesting invariants, but as we shall see in Section 3, at least for $L = \mathbf{Q}$ every generalized Cheeger-Chern-Simons class can be expressed in terms of real second order classes, "classical" Cheeger-Chern-Simons classes and primary classes. For $L = \mathbf{Z}$ the situation is not so clear.

In Section 4 we investigate the relation between generalized Cheeger-Chern-Simons classes and the rational homotopy invariants for foliations studied by Hurder [Hu1], [Hu2] and Hurder-Kamber [Hu-K]. Finally in

Received December 30, 1988.
[1]Work supported in part by the National Science Foundation.